Notes for Lecture 13

Minority, Majority, pn Junction

We treated the carrier equations very generally in the last lecture. Here, we investigate the difference between the minority carrier equation and the majority carrier equation. The minority carrier equation is of immense practical importance.

13.1 Minority Carrier Equation

Let us look at Eqs. (12.28) and (12.29). How would these equations make any difference for minority carrier and majority carrier?

The answer is the drift term, the first term on the RHS of each of those two equations.

In applying these equations, the basic strategy is to consider small quantities in their leading order (linear terms of $O(\varepsilon)$ where ε here just means a small parameter, *not* the dielectric constant!) only in the first approximation. Small quantities in our problem are Δn, Δp, minority carrier density (p_n or n_p), and the internal electric field (if $E \propto \Delta p - \Delta n$, for instance).

Let us, then, assume that the electric field is small, as would be the case for the "quasi-neutral" region of a pn junction, as we shall see later in LN 15. In this case, the drift term is a second order ($O(\varepsilon^2)$) term for the minority carrier, and it can be ignored, while it is a first order ($O(\varepsilon)$) term and should be kept for the majority carrier.

In this section, then, let us deal with the minority carrier case. We will deal with the majority case in the next section.

$$\frac{\partial \Delta n_p}{\partial t} = D_n \frac{\partial^2 \Delta n_p}{\partial x^2} - \frac{\Delta n_p}{\tau_n} \tag{13.1}$$

$$\frac{\partial \Delta p_n}{\partial t} = D_p \frac{\partial^2 \Delta p_n}{\partial x^2} - \frac{\Delta p_n}{\tau_p} \tag{13.2}$$

It makes sense that without the drift term (which involves the coupling of an electric field to charge) these two equations have exactly the same structure. The meaning of the minority equations can be examined using the following two examples.

13.1.1 Minority Carrier Injection I

Suppose that light is turned on suddenly at $t = 0$, and shines on a thin piece of a n-type Si crystal. The light generates electron-hole pairs at a constant rate G_L. Assume that the light is flooding the crystal uniformly, and thus there is no spatial dependence in carrier densities. Let us solve for the time dependence of the minority carrier density p_n.

First of all, some words about the physics of the electron-hole generation by light illumination. As a function of photon energy, the absorption crossection would be very small, when the photon energy is well below the energy gap. For photons with energy well below the energy gap, the semiconductor is essentially transparent to the light. However, as the photon energy approaches the energy gap and go beyond it, there will be a large turn-on of the absorption crossection. This is the so-caled "absorption edge." In this respect, let us look at Fig. T3.20. In that figure, the **absorption coefficient** α is plotted as a function of the wavelength of the photon.

The absorption coefficient is defined as

$$I(x) = I_0 \exp(-\alpha x) \qquad\qquad x \geq 0 \tag{13.3}$$

where the light is incident on a specimen which occupies the region $x \geq 0$ and $I(x)$ is the intensity of light as a function of position inside the specimen, I_0 being the intensity of light that is incident on the specimen. $1/\alpha$ is the **attenuation length** or the **skin depth** of the photon: it measures how deep a photon can penetrate into the specimen.

Looking at Fig. T3.20, notice that $1/\alpha$ is as large as 1 cm at $\lambda = 1.2$ μm, while it becomes as short as 0.1 μm at $\lambda = 0.5$ μm. Why is this? This is due to the absorption edge being in between these two wave lengths. Recall that the energy

versus the wavelength of a photon is given by

$$\hbar\omega \text{ (eV)} = \frac{12398}{\lambda \text{ (Å)}} = \frac{1.2398}{\lambda \text{ } (\mu\text{m})} \tag{13.4}$$

Thus, for the energy gap of Si, ≈ 1.1 eV, we get $\lambda = 1.1$ μm, which explains why the attenuation length becomes so small when λ becomes less than that value.

Since the electron-hole carrier generation is due to the absorption of the photon, we will then have

$$G_L(x) = G_L(0)\, \exp(-\alpha x) \qquad\qquad x \geq 0 \tag{13.5}$$

Coming back to the current example, then, we will assume that the wavelength is slightly lower than that for the energy gap, and the sample dimension is small, so that the light penetrates throughout the crystal. That is, we assume that the sample exists in the region $0 \leq x \leq d$ and that $d \ll 1/\alpha$, and so we can approximate $\exp(-\alpha x) \approx 1$ for any $0 \leq x \leq d$. So, we will simple drop the x dependence in G_L.

The mathematics of this problem is rather simple, with this setup. Without the spatial dependence, the minority carrier equation becomes

$$\frac{\partial \Delta p_n}{\partial t} = -\frac{\Delta p_n}{\tau_p} + G_L \tag{13.6}$$

Note that G_L here is an example of G_{net} as described at the end of the previous lecture note.

This differential equation is an example of a so-called inhomogeneous differential equation, because there is a term G_L, which is not dependent on Δp_n, which is our unknown. The general solution for an inhomogeneous differential equation is the sum of the general solution for the homogeneous version of that differential equation plus any "particular solution." A particular solution is a solution that satisfies the full inhomogeneous equation. For instance, $\Delta p_n = G_L \tau_p$ is a particular solution. Since any particular solution will do, this simplest particular solution is what we will take. Now, let us find the general solution for the homogeneous equation

$$\frac{\partial \Delta p_n}{\partial t} = -\frac{\Delta p_n}{\tau_p} \tag{13.7}$$

While a direct integration by separation of variable is a good way to go for this, I like to keep in mind the following principle.

1. A differential equation is called an n-th order equation, where n refers to the highest order derivative involved. The above example is the first order equation in t, since a first derivative in t is all there is.

2. The general solution for an n-th order differential equation must involve n integration constants. Integration constants are constants that appear as the result of solving ("integrating") the differential equation. They are not related to any quantities in the equation itself, but they are related to the physical conditions such as initial conditions or conserved quantities.

3. Conversely, if a solution with n arbitrary constants are found for a homogeneous differential equation, then that solution is *the* general solution of that homogeneous differential equation.

The third point here gives us a means to find the general solution like the above equation *from experience* but with confidence. Clearly a solution for the above equation is $\Delta p_n = \exp(-t/\tau_p)$, where we simply used the fact that an exponential function is a self-replicating function as far as the differential calculus is concerned. Then, we can easily see that for any arbitrary constant A, $\Delta p_n = A\exp(-t/\tau_p)$ also is a solution to the above equation. Since the first order differential equation has one and only one integration constant, this is *the* general solution, as it has one integration constant A.

Collecting all the information, then, the general solution for the original inhomogeneous equation is

$$\Delta p_n = A\exp(-t/\tau_p) + G_L\tau_p \tag{13.8}$$

How do we determine A? We need to look at the initial condition or the conserved quantities. The initial condition for this problem is that at $t \leq 0$, there must not be any Δp_n, since the light was turned on at $t = 0$. Thus, $\Delta p_n(t = 0) = 0 = A + G_L\tau_p$. Thus, $A = -G_L\tau_p$.

$$\Delta p_n = G_L\tau_p\left[1 - \exp(-t/\tau_p)\right] \tag{13.9}$$

The graph is shown in Figure T3.25. The excess hole density is turned on on a time scale of τ_p and it reaches the steady state value $G_L\tau_p$ for $t \gg \tau_p$. The mathematical behavior shown above is the same as the charging behavior of a capacitor in an RC circuit.

13.1.2 Minority Carrier Injection II

Different from the problem that we just described, now we consider a thick sample and photons with energy slightly higher than the energy gap. This means that the attenuation length of the photon is very short compared to the sample dimension. We also assume that it is very short compared to the diffusion length scale of the

minority carrier that we will be identifying soon (and which were investigated in HW 6.3). This means that we require the photon attenuation length to be much smaller than, say, a micron.

So, let us say that we shine such light at one face of a semiconductor crystal. Again, we assume an n type semiconductor. Note that the Δp at the surface will approach a value $G_L \tau_p$ as we saw in the previous section, if we wait long enough ($t \gg \tau_p$). Let us call this value Δp_{n0}, where 0 here means the surface value.

Question: what would be the distribution of Δp_n everywhere in the crystal in such a steady state reached at $t \gg \tau_p$ after turning on the light?

By the steady state condition, the answer to this question is almost equally easy as in the previous section.

Ignoring the time derivative, we have, for $x > 0$,

$$0 = D_p \frac{\partial^2 \Delta p_n}{\partial x^2} - \frac{\Delta p_n}{\tau_p} \tag{13.10}$$

Note that we do not include G_L term here, since by assumption, the photon generates electron-hole pairs at the surface only ($x = 0$). Identifying the minority carrier diffusion length L as

$$L_p = D_p \, \tau_p \tag{13.11}$$

$$L_n = D_n \, \tau_n \tag{13.12}$$

we can rewrite the above differential equation as

$$\frac{\partial^2 \Delta p_n}{\partial x^2} = \frac{\Delta p_n}{L_p^2} \tag{13.13}$$

This differential equation is of second order, but it is a simple one. The general solution that contains two integration constants are quite easily found from the knowledge of exponential functions[1].

$$\Delta p_n = A \exp\left(-\frac{x}{L_p}\right) + B \exp\left(\frac{x}{L_p}\right) \tag{13.14}$$

Here, we need two conditions to fix both A and B. One condition was already given above – $\Delta p_n = \Delta p_{n0}$ at $x = 0$. Thinking about this experiment, the other can be easily identified: the sample at $x \to \infty$ must not know what goes on at the left end of

[1] In general, for a second order differential equation involving the first derivative as well, a "characteristic equation" can be solved to obtain the general solution. The main idea is always that an exponential function is self-replicating on differentiation.

the sample. If this sounds a bit too much, then you can also say that the hole density certainly should not increase as you go towards the inside of the sample. Either way, the latter condition means that $B = 0$. So, the final solution is

$$\Delta p_n = \Delta p_{n0} \exp\left(-\frac{x}{L_p}\right) \tag{13.15}$$

This means that the minority carrier density is nullified over the length scale of $L_p = \sqrt{D_p \tau_p}$ (cf. Fig. T3.27). The textbook has an apt analogy for L_p: it is the average distance a goldfish (a minority carrier) can travel before being eaten if the water is infested with piranhas (minority carriers).

It should be clearly noted that the square root dependence of L_p on D_p and τ_p results from the physics of random walk, i.e. the physics of diffusion (or Brownian motion; LN11).

If the minority carrier lifetime is on the order of 10 ns, then the diffusion length is on the order of a micron (HW 6.3). However, note that the lifetime can be made much longer, in which case the diffusion length will become much longer as well.

13.1.3 Measurement of Minority Carrier Dynamics

This is the subject of the classic experiment by Haynes and Shockley (HW 7.4).

13.2 Majority Carrier Equation

How about the equation for the majority carriers? From the discussion in the previous lecture note, note that the same dynamics due to τ_p or τ_n exists even when p or n is the majority carrier density. However, this is not the primary dynamics of majority carriers. Simply speaking, the majority carriers carry the electricity by drift, since they outnumber the minority carriers greatly.

We have emphasized above (page 1) that indeed the drift term is not to be neglected in the discussion of majority carriers.

Considering the case with no external electric field, note that any electric field will be generated internally. Here, **Gauss's law** is to be remembered.

$$\nabla \cdot \vec{E} = \frac{\rho}{K_s \epsilon_0} \tag{13.16}$$

Here, ρ is the charge density of the crystal, K_s is the static dielectric constant[2] (at times referred to as ε in my early note!), and ϵ_0 is the vacuum permittivity. Using the potential function V such that $\vec{E} = -\nabla V$, the above equation becomes

$$\nabla^2 V = -\frac{\rho}{K_s \epsilon_0} \tag{13.17}$$

This is the **Poisson equation**.

In our one dimensional case,

$$\frac{\partial E}{\partial x} = \frac{\rho}{K_s \epsilon_0} \tag{13.18}$$

Also, note that

$$\rho(x,t) = e\left[p(x,t) - n(x,t) + N_D(x) - N_A(x)\right] \tag{13.19}$$

where, for once, we write all the position and time dependences of p, n, N_D and N_A explicitly, for clarity.

In case when the equilibrium charge density is zero everywhere (charge neutral uniform device), i.e. if $p_0 - n_0 + N_D - N_A = 0$, then we have

$$\rho(x,t) = e\ \left[\Delta p(x,t) - \Delta n(x,t)\right] \tag{13.20}$$

which means a convenient form of Gauss's law

$$\frac{\partial E}{\partial x} = e\,\frac{\Delta p - \Delta n}{K_s \epsilon_0} \tag{13.21}$$

The drift term for the electron carrier is (from Eq. (12.28)), $\mu_n \frac{\partial (nE)}{\partial x} = \mu_n n \frac{\partial E}{\partial x} + \mu_n E \frac{\partial \Delta n}{\partial x}$, where in the second term we have used the on-going assumption that n_0, p_0 is uniform. Note that the second term is $O(\varepsilon^2)$ and so it should be neglected. Thus, using the above result for Gauss's law, here is the majority carrier equation

$$\frac{\partial \Delta n_n}{\partial t} = \mu_n n_n e\,\frac{\Delta p_n - \Delta n_n}{K_s \epsilon_0} + D_n \frac{\partial^2 \Delta n_n}{\partial x^2} - \frac{\Delta n_n}{\tau_n} \tag{13.22}$$

$$\frac{\partial \Delta p_p}{\partial t} = -\mu_p p_p e\,\frac{\Delta p_p - \Delta n_p}{K_s \epsilon_0} + D_p \frac{\partial^2 \Delta p_p}{\partial x^2} - \frac{\Delta p_p}{\tau_p} \tag{13.23}$$

Notice that while the drift term has different signs for these two equations, its sign is always such that the drift term will try to undo Δn_n or Δp_p, which makes sense.

[2]If we consider a dynamic case, K_s must be changed to an appropriate frequency dependent dielectric constant.

These equations motivate the following time scales and length scales

$$\tau_{n,D} = \frac{K_s \epsilon_0}{\mu_n n_n e} \qquad \text{Dielectric relaxation time (e)} \tag{13.24}$$

$$\tau_{p,D} = \frac{K_s \epsilon_0}{\mu_p p_p e} \qquad \text{Dielectric relaxation time (h)} \tag{13.25}$$

$$\lambda_{n,D} = \sqrt{D_n \tau_{n,D}} \qquad \text{Debye length (e)} \tag{13.26}$$

$$\lambda_{p,D} = \sqrt{D_p \tau_{p,D}} \qquad \text{Debye length (h)} \tag{13.27}$$

The dielectric relaxation time is much shorter than the minority carrier lifetime (τ_n or τ_p), as can be seen from HW 7.3. So, $\Delta n_n/\tau_n$ can be neglected in comparison to $\Delta n_n/\tau_{n,D}$ and similarly $\Delta p_p/\tau_p$ can be neglected in comparison to $\Delta p_p/\tau_{p,D}$. So, here is the final result for the majority equations in the case of no external electric field.

$$\frac{\partial \Delta n_n}{\partial t} = -\frac{\Delta n_n - \Delta p_n}{\tau_{n,D}} + \frac{\lambda_{n,D}^2}{\tau_{n,D}} \frac{\partial^2 \Delta n_n}{\partial x^2} \tag{13.28}$$

$$\frac{\partial \Delta p_p}{\partial t} = -\frac{\Delta p_p - \Delta n_p}{\tau_{p,D}} + \frac{\lambda_{p,D}^2}{\tau_{p,D}} \frac{\partial^2 \Delta p_p}{\partial x^2} \tag{13.29}$$

These equations describe the short time-scale (τ_D) and short length-scale (λ_D) screening action of the majority carriers to ensure the charge neutrality of the crystal beyond these time and length scales (HW 7.3).

13.3 *pn* Junction, Intro

Consider a *pn* junction in equilibrium. We already have learned that there must be a band bending (LN 10). As a convention, we will consider the p part to be on the left side ($x < 0$).

One crucial question is how the charge density ρ behaves right around the point where the p type (N_A) becomes the n type (N_D) (see the activity sheet).

The answer is that ρ abruptly changes with a discontinuity! Furthermore, the ρ on either side of the junction will stay constant for quite a distance before decreasing in magnitude. Why is this?

$\rho = e(p - n + N_D - N_A)$. At the junction N_A suddenly disappears and N_D suddenly appears. At the same time, due to the band bending, we expect that $E_F \approx E_i$ near the junction. This means that near the junction, $p \approx n$. Thus, $\rho \approx e(N_D - N_A)$

near the junction. This is the reason why ρ must have a discontinuity at the junction point. The exponential dependence of p and n on $E_F - E_i$ means that ρ will stay at the value of eN_D on the right side of the junction for an extended range of x. Likewise, it will stay at the value of $-eN_A$ on the left side of the junction for an extended range of x. These extended regions are referred to as the depletion region or the space charge region.

The depletion region is primarily driven by the physics of diffusion, the diffusion of the majority carriers from either side. However, the size of the depletion region is limited by the electric field that is set up by the diffusion itself. Namely, holes move to the right and electrons move to the left at the junction. To the zero-th approximation, a dipole layer is setup. Then, within the depletion region, the electric field will point to the left, limiting the diffusion. That is, drift happens and undoes the effect of diffusion. For this reason, the actual width of the depletion region will be smaller than the diffusion length that we studied above. The balance of the drift and the diffusion determines the physics of the depletion region in equilibrium.