Notes for Lecture 17

Rigid Body

17.1 Rigid body – inertia tensor

The definition of the inertia tensor was given in the previous lecture, but it helps to rewrite it in a more familiar form.

Recall that the vector $\vec{s}_i$ is the position vector of mass m_i, $s_{i,1}$. Let us write its components $s_{i,j}$ as $s_{i,1} = x_i$, $s_{i,2} = y_i$, and $s_{i,3} = z_i$, keeping in mind that these are measured relative to the center of mass, $\vec{R}$. The inertia tensor is then

$$\overleftrightarrow{I} = \sum_i \begin{pmatrix} m_i(y_i^2 + z_i^2) & -m_i x_i y_i & -m_i x_i z_i \\ -m_i x_i y_i & m_i(x_i^2 + z_i^2) & -m_i y_i z_i \\ -m_i x_i z_i & -m_i y_i z_i & m_i(x_i^2 + y_i^2) \end{pmatrix}$$

The matrix given here is for point mass. The summation of matrices mean that the inertia tensor is additive for different parts of the body. Of course, the summation symbol can be brought into the matrix – each component of the matrix should then be summed over. If the mass distribution is continuous, then $\sum_i m_i \to \int dm$.

$$I_{jk} = \int dm \left(\delta_{jk} s^2 - s_j s_k\right)$$

One can recognize that the diagonal terms are what one would expect from the definition given in introductory classes of mechanics: the rotational inertia for a point mass is the mass times distance squared, with distance here meaning the distance between the mass and the rotational axis. For instance I_{11} corresponds to the rotational inertia around the x axis, and so $y_i^2 + z_i^2$ in the expression for I_{11} should be recognized as the distance to the x axis, squared.

Notice that $\overleftrightarrow{I}$ is a real symmetric matrix. This means that, by the well-known theorem in Linear Algebra, it can be diagonalized by an orthogonal matrix $\overleftrightarrow{O}$:

$$\overleftrightarrow{O}^t \, \overleftrightarrow{I} \, \overleftrightarrow{O} \;=\; \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

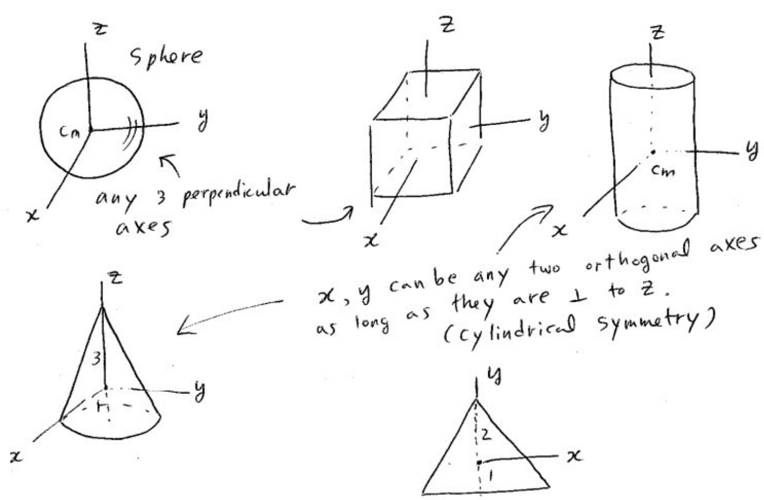
The three column vectors $\vec{O}_j$ of the matrix $\overleftrightarrow{O}$ are, then, the eigenvectors of the inertia tensor $\overleftrightarrow{I}$ with eigenvalue I_j:

$$\overleftrightarrow{I} \, \vec{O}_j \;=\; I_j \vec{O}_j$$

The three orthogonal axes corresponding to the three vectors $\vec{O}_j$ are called the **principal axes of inertia** for the rigid body. The three eigenvalues I_j's are called the **principal moments of inertia**, as opposed to just the **moment of inertia**, which refers to a diagonal element, I_{jj}, of any inertia tensor, a diagonal matrix or not.

Note that the above property of diagonalizability applies to an object with *any* shape. An object may be shaped like a leaf of a tree, a pumpkin or a pebble. However irregular or regular an object's shape is, it is always possible to find three mutually orthogonal axes so that the inertia tensor is diagonal.

For an object with a certain symmetry, it is usually intuitively obvious how to take the principal axes of inertia. The figure below shows some typical examples. If the object is very symmetric (like a sphere, a cube, a cylinder, etc.), then there can be many, often infinite, ways, to take the principal axes of inertia.



For a planar system of particles, it is obvious that a principal axis (call it z) must be perpendicular to the plane, and the other two principal axes (call them x, y) in the plane. Then, for all masses, $z_i = 0$. Therefore, $I_1 = \sum_i m_i(y_i^2 + z_i^2) = \sum_i m_i y_i^2$, and $I_2 = \sum_i m_i(x_i^2 + z_i^2) = \sum_i m_i x_i^2$, while $I_3 = \sum_i m_i(x_i^2 + y_i^2)$.

For a planar system of particles distributed in the xy plane, the following property holds, assuming that xyz axes are the principal axes of inertia.

$$I_3 = I_1 + I_2$$

When the two principal moments of inertia are identical, then we talk of a "**symmetric top**" or a "**symmetric rigid body**." When the three principal momenta of inertia are identical, then we have a case of "**spherical top**." This is not only the case of a sphere, but also the case of a cube.

17.2 Rigid body – pure rotation around any point

In previous sections, we worked strictly within the center of mass frame only.

The center of mass frame is of fundamental importance. As we have seen in the last class, the total linear momentum, the total angular momentum, or the total kinetic energy of a system can *always* be split nicely into two terms, the contribution of a point particle of the total mass M at the position vector $\vec{R}$ and the contribution as viewed from the center of mass reference frame. [For the linear momentum, the latter contribution is always zero, by definition.] This nice separation would not be possible, in general, if any point other than the center of mass is chosen.

However, in some cases, it is useful to consider rotation of a rigid body around a reference point that is not the center of mass. This would be the case if a well-chosen non-center-of-mass reference point makes the description of the whole motion simpler rather than more complicated. When can such a simplification occur? An example is when the reference point can be chosen so that the entire motion can be described as a pure rotation. For instance, the motion of a rolling wheel is a pure rotation around the instantaneous contact point. Such a point of view was very useful for considering that "yo-yo" problem in the first homework. A physical pendulum is another example. It is much simpler to describe the physical pendulum problem as a pure rotation around the pivot point rather than as a motion of the center of mass plus a rotation of the object around its center of mass.

So, let us consider **a pure rotation of a rigid body around a fixed arbitrary point** Q.

For such a pure rotation, the following relations are derived in exactly the same way as we did for the center of mass frame.

$$\vec{L} = \overleftrightarrow{I}_Q \, \vec{\omega}_Q$$
$$T = \frac{1}{2} \vec{\omega}_Q^t \, \overleftrightarrow{I}_Q \, \vec{\omega}_Q$$

Here, we use the subscript Q to note that the inertia tensor and the angular frequency are relative to the point Q. Note that, since we assume a pure rotation, $\vec{L}$ and T are now the *total* angular momentum and the *total* kinetic energy, respectively.

Here, the inertia tensor $\overleftrightarrow{I}_Q$ is given by the exactly same formula as in the previous class, but position vector $\vec{s}_i$ (which was in the center of mass reference frame) now replaced by $\vec{s}_{iQ}$ (which is now referenced to point Q):

$$I_{Q,jk} = \sum_i m_i \left(\delta_{jk} s_{iQ}^2 - s_{iQ,j} s_{iQ,k}\right)$$

In the above expression for the inertia tensor, we can use $s_{iQ,j} = R_{Q,j} + s_{i,j}$, where $\vec{R}_Q$ is the position vector of the center of mass relative to Q, to find the relationship between the inertia tensors $\overleftrightarrow{I}_Q$ and $\overleftrightarrow{I}$, the latter the inertia tensor in the center of mass reference frame as we discussed in the previous lecture.

$$\begin{aligned}
I_{jk,Q} &= \sum_i m_i \left(\delta_{jk} s_{iQ}^2 - s_{iQ,j} s_{iQ,k}\right) \\
&= \sum_i m_i \left(\delta_{jk}\left[\sum_l (R_{Q,l} + s_{i,l})^2\right] - (R_{Q,j} + s_{i,j})(R_{Q,k} + s_{i,k})\right) \\
&= \sum_i m_i \left(\delta_{jk}(R_Q^2 + s_i^2) - R_{Q,j} R_{Q,k} - s_{i,j} s_{i,k}\right) \qquad \because \sum_i m_i s_i = 0 \\
&= I_{jk} + M(\delta_{jk} R_Q^2 - R_{Q,j} R_{Q_k})
\end{aligned}$$

This is a very important result, going by the name of the **parallel axis theorem**. The parallel axis theorem is as yet another example of the "nice separation" for a physical quantity, when the center of mass frame is utilized. Namely, the total inertia tensor is nicely separated into the sum of two terms: (1) the inertia tensor for a point mass M at the position of the center of mass and (2) the "internal" inertia tensor of the body with respect to the center of mass. Such a nice separation occurred for the linear momentum, the angular momentum, or the kinetic energy, as well.

Parallel axis theorem

The inertia tensor $\overleftrightarrow{I}_Q$, defined with respect to an arbitrary point Q, is related to the inertia tensor $\overleftrightarrow{I}$, defined with respect to the center of mass, by the following relation, where $\vec{R}_Q$ is the position vector of the center of mass with respect to point Q.

$$I_{jk,Q} \;=\; M(\delta_{jk}R_Q^2 - R_{Q,j}R_{Q_k}) + I_{jk}$$

or

$$\overleftrightarrow{I}_Q \;=\; M\left(R_Q^2 - \vec{R}_Q\vec{R}_Q^t\right) + \overleftrightarrow{I}$$

In the second form, the term R_Q^2 means a unit matrix times that term. Also, note that $\vec{R}_Q$ is a column vector in the second form, and thus $\vec{R}_Q\vec{R}_Q^t$ is a square matrix (it is the direct product of $\vec{R}_Q$ with itself).

17.3 Euler equations and Eulerian angles

To describe the general motion of a rigid body, it suffices in general to describe the linear motion of the center of mass and the rotational motion of the body around the center of mass. They follow these two general equations of motion.

EOM set 1:

$$\dot{\vec{P}} \;=\; \vec{F}_{ext}$$

$$\dot{\vec{L}}_{int,cm} \;=\; \sum_i \vec{s}_i \times \vec{f}_{ext,i} \;=\; \text{net external torque around the c.m.}$$

The second equation is defined in the center of mass reference frame, which is moving at $\dot{\vec{R}}$ relative to the inertial frame, in which the first equation is defined.

Note that this center of mass reference frame is, in general, a *non*-inertial reference frame in general.

Here, $\vec{f}_{ext,i}$ is the net *external* force that acts on the mass element m_i, and $\vec{F}_{ext} = \sum_i \vec{f}_{ext,i}$. As in the last lecture, $\vec{s}_i$ is the position vector of m_i within the center of mass reference frame.

The above two equations are not just for a rigid body, but for a general system of many particles (cf. next lecture).

The first of the above two equations is merely Newton's second law. When appropriate, it can be replaced by a "rotational" form of Newton's second law.

EOM set 2 (equivalent to set 1):

$$\dot{\vec{L}}_M = \vec{R} \times \vec{F}_{ext}$$

$$\dot{\vec{L}}_{int,cm} = \sum_i \vec{s}_i \times \vec{f}_{ext,i} = \text{net external torque around the c.m.}$$

The second equation is defined in the center of mass reference frame, which is moving at $\dot{\vec{R}}$ relative to the inertial frame, in which the first equation is defined. Note that this center of mass reference frame is, in general, a *non*-inertial reference frame in general.

If the motion under question is a pure rotation, then just one equation is sufficient.

EOM set 3 (can replace set 1 or 2, if a pure rotation):

$$\dot{\vec{L}}_{tot} = \sum_i \vec{r}_i \times \vec{f}_{ext,i} = \text{net external torque}$$

This equation is defined in an inertial frame, in which the motion can be described as a pure rotation.

This equation is all we need for, e.g., the "yo-yo" problem or a physical pendulum.

It is important to note that all of the above equations are valid for any system of particles – whether we have a rigid body or not (it is applicable to a firework, for example). It is just that, for a rigid body, one of these three sets of EOMs are, conveniently, *all* we need to consider, depending on what kind of problem we have. There is, of course, one more convenient equation to remember,

$$\vec{L} = \overleftrightarrow{I}\,\vec{\omega}$$

For the rest of this lecture, we will consider the angular momentum, $\vec{L}$, in the "body reference frame," unless otherwise noted. What is the body reference frame? Imagine that you found the principal axes of a given body. Those principal axes define the body reference frame. We call it a body reference frame, since it is fixed to the body. When the body rolls, the body reference frame rolls the same way.

Now, that type of reference frame sounds pretty tough to deal with. It really is not. When we wrote, $\dot{\vec{L}} =$ net external torque, the above sets of EOM, we *did* consider the possibility that the origin of the reference frame is being accelerated in sets 1 and 2. Even so, by virtue of the center of mass frame, $\dot{\vec{L}} =$ net external torque remains valid. However, we did *not* consider the possibility that the frame rotates just as the rigid body does. This consideration makes us distinguish two frames – the "**body frame**" or the "**rotating frame**" and the "**non-rotating frame**." The body frame, as we defined above, is the frame that is fixed to the body. The non-rotating frame is a reference frame that has the common origin as the body frame, but does not rotate. Instead, its axes have fixed directions.[1]

To summarize the situation, the above sets of EOM were written with the consideration of the non-rotating frame, but for expressing the angular momentum and the angular velocity, we may find the body frame much more useful, since we like to use those useful relations such as $\vec{L} = \overleftrightarrow{I}\,\vec{\omega}$.

So, how do we express the above EOMs in the body frame? We already kind of know how to do this! We consider two frames, a "non-rotating frame" and a "rotating frame". The two frames share the same origin. In the first frame, the xyz axes do not change their orientations, while in the second frame, they do. The above sets of EOM are written with the first type of frame in mind, while now we are finding that the second type of frame is the most convenient for using $\vec{L} = \overleftrightarrow{I}\,\vec{\omega}$. Also, note that the angular velocity of the rotating frame is $\vec{\omega}$ itself. Recall that a vector $\vec{A}$, when

[1] The "non-rotating" frame can have its rotational motion. For instance, imagine a body frame of the Earth and a "non-rotating frame" of the Earth. We take the origin to be the center of the Earth. The "non-rotating frame" is the one that has fixed orientation with respect to the Solar system, while the body frame is the one, additionally, rotating with the spin of the Earth around its axis. However, as the center of the Earth is rotating around the Sun, one might say that the "non-rotating frame" is also, in fact, a rotating one. However, the point is that such rotation is not our main concern here. That rotation is described by the first equation of EOM sets 1 and 2. Our main interest here is EOM set 3, or the second equation of EOM sets 1, 2.

rotated with the angular velocity $\vec{\omega}$, has a rate of change $\dot{\vec{A}} = \vec{\omega} \times \vec{A}$. From this, we get the following relation.

$$\left(\frac{d\vec{A}}{dt}\right)_{nr} = \left(\frac{d\vec{A}}{dt}\right)_{r} + \vec{\omega} \times \vec{A}$$

Here, nr means "non-rotating" and r means "rotating" with the angular velocity $\vec{\omega}$. This equation can be easily understood by considering the difference, $(d\vec{A})_{nr} - \vec{\omega} dt \times \vec{A}$. This difference would be zero if $\vec{A}$ was changing only due to the rotation $\vec{\omega}$. That is, the difference is the $(d\vec{A})_r$.

This consideration leads us to the Euler equation, by taking $\vec{A}$ to be the angular momentum vector, which is expressed as $\overleftrightarrow{I}\,\vec{\omega}$ in the rotating frame.

$$\overleftrightarrow{I}\,\frac{d\vec{\omega}}{dt} + \vec{\omega} \times (\overleftrightarrow{I}\,\vec{\omega}) = \vec{N}_{ext}$$

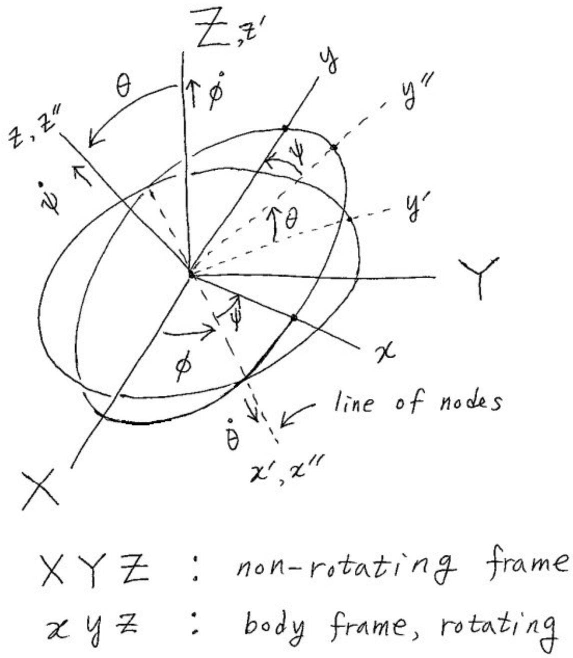
Here, $\vec{N}$ is the torque. Assuming that we are using the principal axes of inertia, (this is what we will assume, unless otherwise noted)

$$I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = N_1$$

plus two more equations by cyclic permutations ($123 \to 231 \to 312$).

These are the so-called **Euler equations**.

The three degrees of freedom associated with the rotation of a rigid body can be expressed as **Eulerian angles**, ϕ, θ, ψ, as defined in the following diagram.



The diagram shows the three rotations that transforms the non-rotating frame ("nr") XYZ to the "body frame" or the rotating frame ("r") xyz. The two intermediate frames are defined as the $x'y'z'$ frame and the $x''y''z''$ frame. Here, ϕ and θ can be thought of as the standard angles from the spherical coordinate system. They specify the instantaneous direction of the z axis of the body frame. The rotation matrices are elementary, and they are easy to write down [just remember that a coordinate transformation matrix can be constructed simply by collecting column vectors, each of which corresponds to the representation of the old unit vector in the new coordinate system], when necessary (see textbook). By inspection of the above diagram, we can express the angular velocity in the rotating frame.

$$\begin{aligned}
\omega_1 &= \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\
\omega_2 &= \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\
\omega_3 &= \dot{\psi} + \dot{\phi}\cos\theta
\end{aligned}$$

$\omega_1, \omega_2, \omega_3$ are the components of the angular velocity as seen in the body reference frame xyz. In particular, ω_3 corresponds to how fast the body is spinning about the z axis.

17.4 Free top

We consider a rigid body under zero external torque ("free top"). In this case, the energy

$$E = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$$

as well as the angular momentum

$$\vec{L} = I_1\omega_1\hat{x} + I_2\omega_2\hat{y} + I_3\omega_3\hat{z}$$

are constants of motion.

What is immediately noticeable is that, in general, the angular momentum $\vec{L}$ does not point in the same direction as $\vec{\omega}$, if I_i $(i = 1, 2, 3)$'s are not all identical. This is for example why the "balancing of a wheel" is important. When a wheel is not balanced properly, a car can suffer from unpleasant vibrations at high speeds. This is because the angular momentum of the wheel is slightly off axis. If the wheel is to keep rotating with the angular velocity parallel to the axle, then the angular momentum must be changing. Namely, this is not a possible motion for a "free top," considering the wheel as a top. So, as the wheel rotates, it *requires* that there be a torque that is consistent with the angular momentum change. That requirement may not be met, since the axle is not designed to provide such a torque at high speeds. Sure, there is that torque coming from the engine to rotate the axle. But that torque is aligned to the axle, not perpendicular to it, as the unbalanced wheel would require. So, what happens is that then angular velocity cannot be nicely aligned to the axle. This results in various joints of the car getting some "beating," which is the source of all those unpleasant vibrations and noises.

If a wheel has a nearly spherical shape, then that wheel balance problem will be less critical. For a spherical top, $I_1 = I_2 = I_3$, and $\vec{\omega}$ will always be in the same direction as $\vec{L}$.

Let us consider a free spherical top. $I_1 = I_2 = I_3$ means that, from Euler's equation, $\dot{\omega}_1 = \dot{\omega}_2 = \dot{\omega}_3 = 0$. This seems right. A soccer ball that spins will keep spinning in the same direction. That direction is the direction of the angular momentum.

For an arbitrary free top, this is not true in general, as we will see shortly. However, it can be true, if it is given an initial angular velocity along one of the principal axes. Namely, for any shape of a free rigid body, if it is initially rotated around one of its principal axes, then it will keep rotating with the same angular velocity forever. This is because, according to the Euler equations, $\dot{\omega}_i = 0$ for all i, if the two angular velocity components are zero for a free top. This is because the second term in Euler equations appear as a product of angular velocity components. In this special case, $\vec{L}$ will be parallel to $\vec{\omega}$ for any top.

This does not necessarily mean that these rotations are stable. Just as a particle on top of a circular hill is unstable to any perturbation, some of these rotations can be unstable. A simple demo with any rectangular shape can show that not all rotations are stable. How do we analyze this? One way to do it to geometrically examine the traces ("polhodes") of the angular momentum vector as the body rotates (see Landau section 37, e.g.). Here, we use a more analytical approach, using Euler's equations, directly.

Consider

$$\vec{\omega} = \omega_1 \hat{x} + \lambda \hat{y} + \mu \hat{z}$$

where $\omega_1 \gg \lambda, \mu$. The Euler equations to solve are

$$\begin{aligned} I_1 \dot{\omega}_1 &= (I_2 - I_3)\lambda\mu \\ I_2 \dot{\lambda} &= (I_3 - I_1)\mu\omega_1 \\ I_3 \dot{\mu} &= (I_1 - I_2)\lambda\omega_1 \end{aligned}$$

We will analyze these equations up to linear order in λ, μ. That means that the right hand side of the first equation can be ignored, giving $\omega_1 =$ constant. Then, combining the second and the third equations, we get

$$\begin{aligned} \ddot{\lambda} &= \frac{(I_3 - I_1)(I_1 - I_2)}{I_2 I_3}\omega_1^2 \lambda \\ \ddot{\mu} &= \frac{(I_3 - I_1)(I_1 - I_2)}{I_2 I_3}\omega_1^2 \mu \end{aligned}$$

Defining

$$\Omega^2 = \frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}\omega_1^2$$

, we note that both λ and μ satisfy the equation of the form $\ddot{q} = -\Omega^2 q$. If Ω^2 is indeed positive, then the solution is an oscillatory one with the angular frequency given by $Omega$. **This is the situation for a stable rotation**. However, if Ω^2 is negative, then Ω becomes complex, and the solution is $q = A\exp(-|\Omega|t) + B\exp(|\Omega|t)$. **This is then the situation for an unstable rotation**. So, when $I_2 < I_1 < I_3$, we get an unstable rotation. If an object is rotated around the principal axis for which the rotational inertia is the largest or the smallest, then the rotation is stable. However, if it is rotated around the principal axis for which the rotational inertia is neither the maximum nor the minimum, then that rotation is unstable.

17.5 Free symmetric top

Let us consider a free symmetric top.

Assume $I_1 = I_2 = I$. Then, we get the following results from Euler's equations.

$$\begin{aligned} I\dot{\omega}_1 &= (I - I_3)\omega_2\omega_3 \\ I\dot{\omega}_2 &= (I_3 - I)\omega_3\omega_1 \\ I_3\dot{\omega}_3 &= 0 \end{aligned}$$

These equations are analogous to what we just considered for an arbitrary top. Except that these equations are valid for any values of ω_i.

The first thing to notice is that ω_3 is constant. For the other two, we get

$$\begin{aligned} \ddot{\omega}_1 &= -\frac{(I - I_3)^2}{I^2}\omega_3\omega_1 \\ \ddot{\omega}_2 &= -\frac{(I - I_3)^2}{I^2}\omega_3\omega_2 \end{aligned}$$

So, setting (we assume that $\omega_3 > 0$),

$$\Omega = \frac{|I - I_3|}{I}\omega_3$$

we conclude that ω_1 and ω_2 are both solutions corresponding to a simple harmonic oscillation with the angular frequency Ω.

By choosing the origin of time properly, we can take

$$\omega_1 = \omega\cos(\Omega t)$$

What is ω_2? Note, from Euler equations, that ω_2 is proportional to $\dot{\omega}_1$. Two cases need to be distinguished.

Let us consider the "prolate" case ($I > I_3$), like a thin rod.

In this case, $\Omega = \frac{I - I_3}{I}\omega_3$, and $\omega_2 = \dot{\omega}_1/\Omega$.

$$\omega_2 = -\omega\sin(\Omega t) \quad \text{Prolate, } I > I_3$$

In the "oblate" case ($I < I_3$), like a thin disk, we get

$$\omega_2 = \omega\sin(\Omega t) \quad \text{Oblate, } I < I_3$$

To summarize, the angular velocity has a constant z component, and its x, y components show a circular motion. So, $\vec{\omega}$ traces a cone ("body cone"). We assumed above that ω_3 is positive. The angular velocity vector rotates clockwise for a prolate body, and counter-clockwise for an oblate body.

This is the view from the body frame.

Let us ask what $\dot{\theta}$, $\dot{\phi}$ (Euler angles) are. The following consideration is helpful.

First of all, $\vec{L}$ must be a constant vector in the non-rotating frame, as there is no external torque. Let us define, then, the Z axis of the non-rotating frame be the direction of $\vec{L}$. Second, the z axis of the body frame makes θ from the Z axis. Due to the symmetric nature of the top, the xy axes can be taken as any two axes perpendicular to the z axis. For a given instant, we can choose the x axis to be in the plane formed by $\vec{L}$ and the z axis (i.e., we choose $\psi = \pi/2$). Then, it is easy to deduce that $\vec{\omega}$ must be in the same plane, as $L_y = 0$. Now, note that any point of the z axis will move according to $d\vec{r}/dt = \omega \times \vec{r}$. Namely, $d\vec{r}$ is perpendicular to the plane of consideration. Thus, $\theta =$ constant.

$$\dot{\theta} \;=\; 0 \quad \text{(no nutation)}$$

Using this, and using the fact that $\psi = \pi/2$, we get $\omega_1 = \dot{\phi}\sin\theta$. Since this must be equal to $L\sin\theta/I$, we get

$$\dot{\phi} \;=\; L/I \quad \text{(precession)}$$

So, the motion as viewed from the non-rotating frame is that of a precession of the z axis of the body by the angular frequency L/I. The z axis of the body traces a cone around the angular momentum vector.

Now, let us consider what is known as "Feynman problem." Consider a flat disk, at a very small θ value. How fast does a disk wobble (precess) compared to its spin? The speed of the spin is given by $\omega_3 = L\cos\theta/I_3$. For a disk, $I_3 = 2I$, and using $\cos\theta \approx 1$ for small θ, we get

$$\omega_3 \;\approx\; L/(2I) \;=\; \dot{\phi}/2 \quad \text{(Feynman's dining hall problem)}$$

So, a disk precesses twice as fast.

17.6 Fast symmetric top

Let us now consider the case of an external torque acting on the top due to external force, gravity and possibly other force such as friction or normal force.

For a problem such as a symmetric top under gravity with a fixed end point (section 11.11 of the textbook) it is possible to solve the problem in an analogous manner as we did for the bead on a rotating circular wire. In this case, the canonical momenta p_ψ and p_ϕ are conserved, as well as the energy, and thus, the problem can be turned into an effective one dimensional one. The solution consists of an oscillation in the θ value, which is called *nutation*. So, in this case, the symmetric top goes through both precession ($\dot{\phi}$) and nutation ($\dot{\theta}$). The mathematical details are left for your reading (11.11 of textbook). Also, using the effective potential that is derived in this approach, it is possible to discuss the stability of a symmetric top that is spinning vertically. A vertical symmetric top is stable if $\omega_3^2 > 4IMgl/I_3^2$, where $I = I_1 = I_2$ as before (here, I is the rotational inertia around the fixed point, which is not the center of mass), and l is distance between the center of mass and the fixed point.

Here, we instead examine a slightly different problem, a fast top. By fast top, we mean that the top is rotating so fast so that the gravity or any other external force can be treated as a small perturbation.

Then, in the zeroth order, a fast symmetric top is a free top, with a fixed angular momentum vector $\vec{L}$. The z axis of the top rotates around $\vec{L}$ with the angular velocity $\vec{\Omega} = \vec{L}/I$. Here, the direction of $\vec{L}$ is not vertical in general. $\vec{\Omega}$ is a fast rotation of the z axis around the angular momentum vector, and is better called *nutation* than precession.

For a fast top, the effect of an external torque is then to change the direction of $\vec{L}$ at a rate much smaller than Ω. Let us consider the case that the end point of the fast top is fixed and the gravity is acting on the top. Then, the torque on the body is given by

$$d\vec{L}/dt = M\vec{R} \times \vec{g}$$

Note that $\vec{R}$ and $\vec{L}$ are not parallel at any given time due to the nutation $\vec{\Omega}$. However, they would be parallel to each other if the fast nutation is averaged over. Doing this average, we get

$$d\vec{L}/dt = Ml\vec{L} \times \vec{g} \cos\alpha/L$$

where l is the distance between the center of mass and the fixed point, and α is the angle between the z axis of the body and $\vec{L}$. This equation means that $\vec{L}$ is precessing with

$$\vec{\Omega}_{pr} \;=\; -Ml\vec{g}\cos\alpha/L$$

This explains how a bicycle wheel gyroscope works.