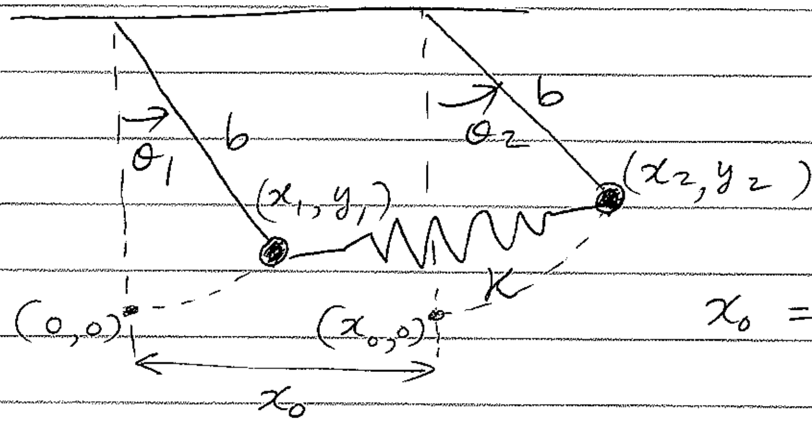
Ex.



x_0 = natural length of spring

$$(x_1, y_1) = (b S_{\theta_1}, b(1 - C_{\theta_1}))$$

$$(x_2, y_2) = (x_0 + b S_{\theta_2}, b(1 - C_{\theta_2}))$$

distance between two masses

$$= \sqrt{\{x_0 + b(S_{\theta_2} - S_{\theta_1})\}^2 + \{b(C_{\theta_1} - C_{\theta_2})\}^2}$$

$$= \sqrt{x_0^2 + 2bx_0(S_{\theta_2} - S_{\theta_1}) + b^2(S_{\theta_2} - S_{\theta_1})^2 + b^2(C_{\theta_1} - \theta_2)^2}$$

$$\simeq x_0\left\{1 + \frac{b}{x_0}(S_{\theta_2} - S_{\theta_1})\right\}$$

We need to keep terms up to $x_0 + \begin{pmatrix}\text{linear in} \\ \theta_1, \theta_2\end{pmatrix}$

Δx of the spring

$$= b(S_{\theta_2} - S_{\theta_1})$$

Δx is linear in (θ_1, θ_2) $\therefore U \propto$ quadratic

$$U = \frac{1}{2}k b^2 (S_{\theta_2} - S_{\theta_1})^2 + mgb(1 - C_{\theta_1}) + mgb(1 - C_{\theta_2})$$

$$\simeq \frac{1}{2} k b^2 (\theta_2 - \theta_1)^2 + \frac{1}{2} mgb(\theta_1^2 + \theta_2^2)$$

$$T = \frac{1}{2} m b^2 \left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right)$$

$$\overleftrightarrow{M} = \begin{bmatrix} b^2 m & 0 \\ 0 & b^2 m \end{bmatrix} \qquad \overleftrightarrow{A} = \begin{bmatrix} mgb + kb^2 & -kb^2 \\ -kb^2 & mgb + kb^2 \end{bmatrix}$$

$$\overleftrightarrow{A}\,\vec{T_i} = \omega_i^2\, \overleftrightarrow{M}\,\vec{T_i}$$

$$\left|\overleftrightarrow{A} - \omega^2 \overleftrightarrow{M}\right| = \begin{vmatrix} mgb + kb^2 - \omega^2 b^2 m & -kb^2 \\ -kb^2 & mgb + kb^2 - \omega^2 b^2 m \end{vmatrix} = 0$$

$$(mg + kb - \omega^2 bm)^2 = \pm kb$$

$$\omega^2 bm = mg + kb \pm kb$$

$$= mg \quad \text{or} \quad mg + 2kb$$

$$\omega^2 = \frac{g}{b} \quad \text{or} \quad \frac{g}{b} + \frac{2k}{m}$$

(soft) (hard)

$$\omega_1^2 = \frac{g}{b} \Rightarrow \overleftrightarrow{A} - \omega^2 \overleftrightarrow{M} = \begin{bmatrix} kb^2 & -kb^2 \\ -kb^2 & kb^2 \end{bmatrix}$$

$$\therefore \vec{T_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$ up to a multiplicative const.

$$\omega_2^2 = \frac{g}{b} + \frac{2k}{m} \Rightarrow \overleftrightarrow{A} - \omega^2 \overleftrightarrow{M} = \begin{bmatrix} -kb^2 & -kb^2 \\ -kb^2 & -kb^2 \end{bmatrix}$$

$$\therefore \vec{T_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$ "

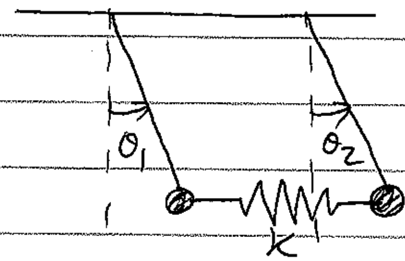
$\vec{T}_1 \eta_1$

For the soft mode

$$\omega_1^2 = \frac{g}{b} \Rightarrow \vec{\theta} = \overleftrightarrow{T}\begin{bmatrix}\eta_1\\ 0\end{bmatrix}$$

$$\theta_1 = \theta_2 = \eta_1$$

motion for $\rightarrow$ η_1

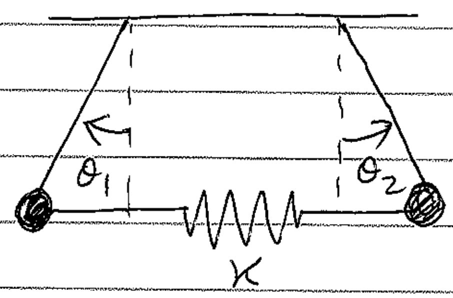


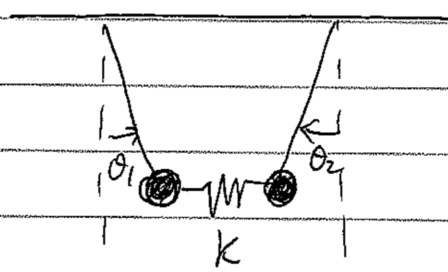
No spring force.

$\omega_1^2 = \frac{g}{b}$ ← just like simple pendulum.

For the hard mode, $\omega_2^2 = \frac{g}{b} + 2\frac{K}{m}$

motion for $\rightarrow$ η_2





$$\vec{\theta} = \overleftrightarrow{T}\begin{bmatrix}0\\ \eta_2\end{bmatrix} = \vec{T}_2\eta_2$$

$$= \begin{bmatrix}\eta_2\\ -\eta_2\end{bmatrix}$$

$$\theta_1 = -\theta_2 = \eta_2$$

Note that in this case if g is "turned off" then $\omega_2^2 = \frac{2K}{m}$ ... why?

Internal motion of 2 body with potential $U = \frac{1}{2}Kx^2$, reduced mass $= \frac{m}{2}$

$$\therefore \omega^2 = \frac{K}{\frac{m}{2}} = \frac{2K}{m}$$

This problem is essentially identical with the 1st problem.

1st example = example of L14 note, I mean

To complete the solution, ask what are η_1, η_2 ?

(normal coordinates)

$$\vec{x} = \overleftrightarrow{T}\vec{\eta} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ \vec{T}_1 & \vec{T}_2 \end{array}$$

$$\vec{\eta} = \overleftrightarrow{T}^{-1}\vec{x} \qquad \overleftrightarrow{T}^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\therefore \eta_1 = \frac{1}{2}(x_1 + x_2), \quad \eta_2 = \frac{1}{2}(x_1 - x_2)$$

<u>Note</u>. If we defined $\overleftrightarrow{T} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

($\because$ a multiplicative constant at our disposal for <u>each</u> column vector)

then $\overleftrightarrow{T}$ is an orthogonal matrix!

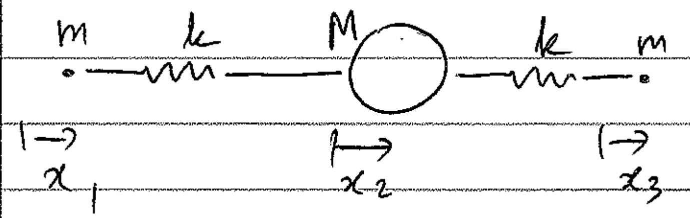
$$\overleftrightarrow{T}^{-1} = \overleftrightarrow{T}^{t} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\vec{x} = \overleftrightarrow{T}\vec{\eta} \Rightarrow \begin{aligned} x_1 &= \frac{1}{\sqrt{2}}(\eta_1 + \eta_2) \\ x_2 &= \frac{1}{\sqrt{2}}(\eta_1 - \eta_2) \end{aligned}$$

$$\vec{\eta} = \overleftrightarrow{T}^{-1}\vec{x} \Rightarrow \begin{aligned} \eta_1 &= \frac{1}{\sqrt{2}}(x_1 + x_2) \\ \eta_2 &= \frac{1}{\sqrt{2}}(x_1 - x_2) \end{aligned}$$

$\overleftrightarrow{T}$ is <u>not</u> always orthogonal, though. (See next.)

§. Longitudinal modes of a linear molecule (e.g. CO_2)
--- Classical Mechanics model (crude but <u>very</u> useful.)



$$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_3^2 + \frac{1}{2} M \dot{x}_2^2$$

$$U = \frac{1}{2} k (x_1 - x_2)^2 + \frac{1}{2} k (x_3 - x_2)^2$$

$$\overleftrightarrow{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{bmatrix} \qquad \overleftrightarrow{A} = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}$$

$$\left| \overleftrightarrow{A} - \omega_i^2 \overleftrightarrow{M} \right| = 0$$

$$\Rightarrow \left| \frac{\overleftrightarrow{A}}{k} - \frac{\omega_i^2}{k} \overleftrightarrow{M} \right| = 0$$

Let's define $\lambda = \frac{\omega_i^2}{k} m$

$$\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\frac{M}{m}\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left\{ \left(2-\frac{M}{m}\lambda\right)(1-\lambda) - 1 \right\} + \left\{ (-1)\cdot(1-\lambda) \right\} = 0$$

$$(1-\lambda)\left\{ \left(2-\frac{M}{m}\lambda\right)(1-\lambda) - 2 \right\} = 0$$

$$(1-\lambda)\cdot\left\{ \frac{M}{m}\lambda^2 - \left(2+\frac{M}{m}\right)\lambda \right\} = 0$$

$$\underline{\lambda = 1, \; 0, \; \frac{M+2m}{M}} \; ✓$$

(1) $\lambda = 0 \dashrightarrow \omega_1^2 = 0$ $\qquad x_1 = x_2 = x_3 \qquad \vec{T}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

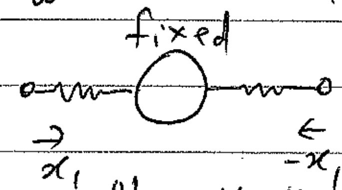
$$\underbrace{\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix}}_{\vec{T}_1} = 0 \rightarrow \begin{matrix} u = v \\ v = \frac{1}{2}(u+w) \\ v = w \end{matrix} \rightarrow \underline{u = v = w}$$

translation!!

no restoring force!

(2) $\lambda = 1 \dashrightarrow \omega_2^2 = \frac{k}{m}$

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 2-\frac{M}{m} & -1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = 0 \rightarrow \begin{matrix} v = 0 \\ u = -w \end{matrix} \qquad \vec{T}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$



"breathing" mode

$$\vec{x} = \vec{T}_2 \eta_2 = \begin{bmatrix} \eta_2 \\ 0 \\ -\eta_2 \end{bmatrix}$$

$\therefore x_1 = -x_3, \; x_2 = 0$

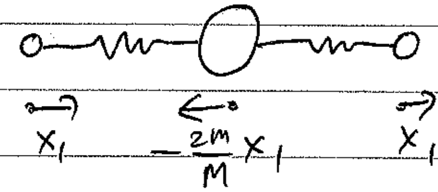
(3) $\lambda = \frac{M+2m}{M} \dashrightarrow \omega_3^2 = \frac{M+2m}{Mm} \cdot k$

$$\begin{bmatrix} -\frac{2m}{M} & -1 & 0 \\ -1 & -\frac{M}{m} & -1 \\ 0 & -1 & -\frac{2m}{M} \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = 0 \rightarrow \begin{matrix} v = -\frac{2m}{M}u \\ v = -\frac{2m}{M}w \end{matrix} \rightarrow u = w$$

$$\vec{T}_3 = \begin{bmatrix} 1 \\ -\frac{2m}{M} \\ 1 \end{bmatrix}$$

$$\vec{x} = \vec{T}_3 \eta_3 = \begin{bmatrix} \eta_3 \\ -\frac{2m}{M}\eta_3 \\ \eta_3 \end{bmatrix}$$

$$\underbrace{x_1 = x_3 \qquad x_2 = -\frac{2m}{M}x_1}_{\text{meaning?}}$$



x_1 and x_3 in phase

while $\underset{\text{(center of mass)}}{CM} = \frac{m(x_1+x_3)+x_2M}{2m+M} = \frac{2mx_1 - 2mx_1}{2m+M} = 0$

Now, to complete the solution, we ask

what is $\vec{\eta} = ?$

We need to find $\overleftrightarrow{T}^{-1}$. A little tricky...

$$\overleftrightarrow{T} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -\frac{2m}{M} \\ 1 & -1 & 1 \end{bmatrix}$$

$\uparrow \quad \uparrow \quad \uparrow$

$\vec{T}_1 \quad \vec{T}_2 \quad \vec{T}_3$

$\Leftarrow$ Cannot be made into an orthogonal matrix!

$\vec{T}_3 \cdot \vec{T}_1 \neq 0$

We have to invert $\overleftrightarrow{T}$ by brute force...

$$\overleftrightarrow{T}^{-1} = (-) \cdot \frac{M}{2M+4m} \cdot \begin{bmatrix} -\frac{2m}{M} & -2 & -\frac{2m}{M} \\ -1-\frac{2m}{M} & 0 & \frac{2m}{M}+1 \\ -1 & 2 & -1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} \eta_1 = \dfrac{1}{M+2m}\left(m x_1 + M x_2 + m x_3\right) \\ \eta_2 = \dfrac{1}{2}\left(x_1 - x_3\right) \\ \eta_3 = \dfrac{1}{2(M+2m)}\left(x_1 + x_3 - 2x_2\right) \end{cases}$$

$\nwarrow$ CM coordinate

Note

1. Why was it not possible to turn $\overleftrightarrow{T}$ into an orthognal matrix by normalizing each column vector?

<u>Ans.</u> $\overleftrightarrow{M}$ was not a constant times an identity matrix.

2. In some problems, λ (or ω_i^2) may be degenerate --- what to do about $\vec{T}_i$ in those cases?

<u>Ans.</u> Indeed, this happens. E.g. in homework problems.

$$\overleftrightarrow{A} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad \overleftrightarrow{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

up to a const   up to a const

$\Rightarrow \quad \lambda = 1$ or 4

$1 \rightarrow$ degenerate (double root)

$\Rightarrow$ For $\lambda = 1$, all we get for $\vec{T}_c = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$ is

$\underline{u + v + w = 0}$.

It is a plane!

You can pick two orthogonal vectors in that plane, e.g. $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ as $\vec{T}_1$ and $\vec{T}_2$

You are free to choose any two linearly independent vectors for a doubly degenerate eigenvalue.