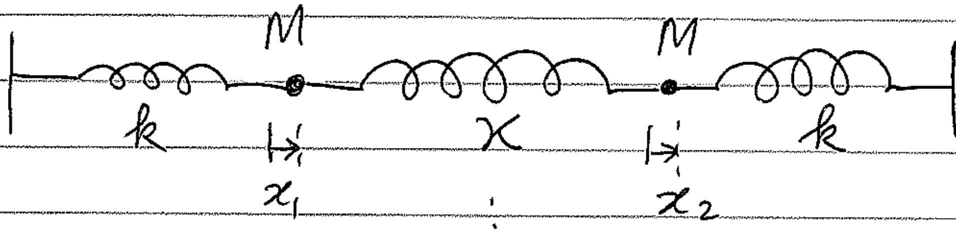
## §. Coupled Oscillators




"kappa"

① **Newton method**

$$M\ddot{x}_1 = -\kappa(x_1 - x_2) - k x_1$$
$$M\ddot{x}_2 = -\kappa(x_2 - x_1) - k x_2 \qquad (*)$$

Assume
$$x_1 = u\, e^{i\omega t}$$
$$x_2 = v\, e^{i\omega t} \qquad (**)$$

(i) We are now in the complex plane. When done, we can go to the real axis by take $Re\{x_1\}$ and $Re\{x_2\}$.

(ii) By assuming $e^{i\omega t}$, we are conjecturing that a SHO like motion exists. We will justify this more generally, soon.

$(**) \Rightarrow (*)$ :
$$-M\omega^2 u = -\kappa(u-v) - ku$$
$$-M\omega^2 v = -\kappa(v-u) - kv$$

$$\Rightarrow \quad (k+\kappa - M\omega^2)\, u - \kappa v = 0$$
$$-\kappa u + (k+\kappa - M\omega^2)\, v = 0$$

$$\Rightarrow \begin{bmatrix} k+\kappa - M\omega^2 & -\kappa \\ -\kappa & k+\kappa - M\omega^2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0$$

"Eigen-value equation"

"~~Characteristic~~ equation"

Non-trivial ($u \neq 0$ and $v \neq 0$) solution is possible only when

$$\begin{vmatrix} k+K-M\omega^2 & -K \\ -K & k+K-M\omega^2 \end{vmatrix} = 0$$

characteristic eq.
Secular eq.

$$\Rightarrow \quad (k+K-M\omega^2)^2 - K^2 = 0$$

$$M\omega^2 = k+K \pm K$$

$$\omega = \sqrt{\frac{k}{M}} \quad \text{or} \quad \sqrt{\frac{k+2K}{M}}$$

"soft" mode   "hard" mode

"Eigenvalues"
Charcteristic frequencies
Eigenfreq's

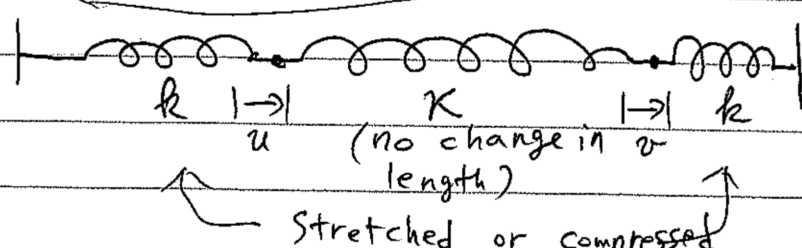
Plug the eigenvalues into the eigenvalue eq.

$\Rightarrow$ For ① $\omega = \sqrt{\frac{k}{M}}$, $\begin{bmatrix} K & -K \\ -K & K \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = 0$

soft mode

$\therefore \quad u = v$

"in-phase" →

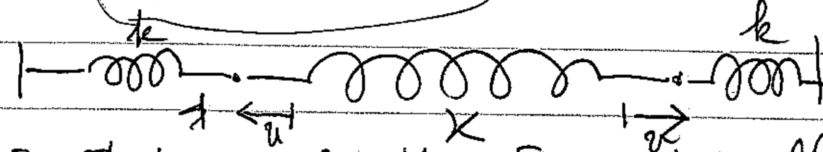



② $\omega = \sqrt{\frac{k+2K}{M}}$, hard mode $\begin{bmatrix} -K & -K \\ -K & -K \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = 0$

"breathing mode"

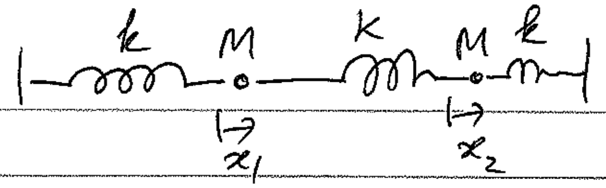
$\therefore \quad u = -v$

"out-of-phase"




All springs change lengths. Symmetrically.

② <u>Lagrangian method</u>




$$T = \frac{1}{2} M \left( \dot{x}_1^2 + \dot{x}_2^2 \right)$$

$$U = \frac{1}{2} k\, x_1^2 + \frac{1}{2} K (x_1 - x_2)^2 + \frac{1}{2} k\, x_2^2$$

~~(Hooke's law)~~

From here, we can derive the EOMs for $x_1$ and $x_2$, ---, the same deal as what we've done with the Newton's method. But, wait, let's do the following ...

$$T = \frac{1}{2} [\dot{x}_1 \ \dot{x}_2] \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

$$U = \frac{1}{2} [x_1 \ x_2] \begin{bmatrix} k+K & -K \\ -K & k+K \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

How did this happen??

$$U = \frac{1}{2} k x_1^2 + \frac{1}{2} K x_1^2 - K x_1 x_2 + \frac{1}{2} K x_2^2 + \frac{1}{2} k x_2^2$$

$$= \frac{1}{2} (k+K) x_1^2 - K x_1 x_2 + \frac{1}{2} (k+K) x_2^2$$

$$= \frac{1}{2} \sum_{i,j=1}^{2} x_i\, x_j\, A_{ij} \quad \Rightarrow \quad A_{11} = k+K, \quad A_{22} = k+K, \quad A_{12} = -K$$

$$= \frac{1}{2} [x_1 \, x_2] \begin{bmatrix} A_{ij} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

So, $L = \frac{1}{2}\dot{\vec{x}}^t \overleftrightarrow{M} \dot{\vec{x}} - \frac{1}{2}\vec{x}^t \overleftrightarrow{A} \vec{x}$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is a column vector

$\vec{x}^t = [x_1 \;\; x_2]$ is a row vector

$\overleftrightarrow{M} = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}$ and $\overleftrightarrow{A} = \begin{bmatrix} k+K & -K \\ -K & k+K \end{bmatrix}$

(mass "tensor") (stiffiness "tensor")

are square matrices.

Note

a) $\overleftrightarrow{A}$ and $\overleftrightarrow{M}$ are symmetric : $\overleftrightarrow{A} = \overleftrightarrow{A}^t$, $\overleftrightarrow{M} = \overleftrightarrow{M}^t$

b) $\overleftrightarrow{M}$ is positive definite.

That is $\dot{\vec{x}}^t \overleftrightarrow{M} \dot{\vec{x}} > 0$ unless $\dot{\vec{x}} = 0$ in which case $\dot{\vec{x}}^t \overleftrightarrow{M} \dot{\vec{x}} = 0$

$\frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2)$ --> obviously positive definite in this problem.

! Big Question !

Consider a coordinate transform (or change of basis) $\vec{x} = \overleftrightarrow{T}\vec{\eta}$ $\quad \vec{\eta} = \overleftrightarrow{T}^{-1}\vec{x} = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$

$\overleftrightarrow{T} = \overleftrightarrow{a}$ in the book

$\overleftrightarrow{T}$ : invertible ~~square~~ matrix

(often orthogonal, but we don't need to assume orthogonal here)

$L = \frac{1}{2}\dot{\vec{\eta}}^t \{\overleftrightarrow{T}^t \overleftrightarrow{M} \overleftrightarrow{T}\} \dot{\vec{\eta}} + \frac{1}{2}\vec{\eta}^t \{\overleftrightarrow{T}^t \overleftrightarrow{A} \overleftrightarrow{T}\} \vec{\eta}$

Can we find $\overleftrightarrow{T}$ s.t.

$\overleftrightarrow{T}^t \overleftrightarrow{M} \overleftrightarrow{T} = \begin{bmatrix} m_1^* & 0 \\ 0 & m_2^* \end{bmatrix}$ and $\overleftrightarrow{T}^t \overleftrightarrow{A} \overleftrightarrow{T} = \begin{bmatrix} k_1^* & 0 \\ 0 & k_2^* \end{bmatrix}$ ??

## § Motivation for the big Q

If the answer is yes, then

$$L = L_1 + L_2$$

$$L_1 = \frac{1}{2} m_1^* \dot{\eta}_1^2 - \frac{1}{2} k_1^* \eta_1^2 \quad \text{--- a SHO} \quad \omega_1^2 = \frac{k_1^*}{m_1^*}$$

$$L_2 = \frac{1}{2} m_2^* \dot{\eta}_2^2 - \frac{1}{2} k_2^* \eta_2^2 \quad \text{--- a SHO} \quad \omega_2^2 = \frac{k_2^*}{m_2^*}$$

The coupled oscillator problem becomes two completely decoupled SHO problem!!

## § But, is the answer yes??

Yes! Yes!

$\eta_1, \eta_2$ defined by such $\overleftrightarrow{T}$ are called "normal coordinates" or "normal modes".

Note: In general, the answer is yes no matter how many particles are involved and no matter how the coupling is made. As long as $\overleftrightarrow{A}, \overleftrightarrow{M}$ are symmetric and $\overleftrightarrow{M}$ positive definite, the answer to the big Q is yes!!

$$\vec{\eta} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \end{bmatrix} \leftarrow \text{many normal coordinates} \leftarrow \text{\# of degrees of freedom}$$

Just math skip if you like

How does the answer "yes" come out?
Let us go back to the 2D problem.
We want

$$\overleftrightarrow{T}^t \overleftrightarrow{M} \overleftrightarrow{T} = \begin{bmatrix} m_1^* & 0 \\ 0 & m_2^* \end{bmatrix} \quad \cdots ①$$

$$\overleftrightarrow{T}^t \overleftrightarrow{A} \overleftrightarrow{T} = \begin{bmatrix} k_1^* & 0 \\ 0 & k_2^* \end{bmatrix} \quad \cdots ②$$

① $\cdots$ $$\overleftrightarrow{T}^t \overleftrightarrow{M} \overleftrightarrow{T} \begin{bmatrix} m_1^{*-1} & 0 \\ 0 & m_2^{*-1} \end{bmatrix} = \overleftrightarrow{1} \quad \cdots ①'$$

①' on ② just before $\begin{bmatrix} k_1^* & 0 \\ 0 & k_2^* \end{bmatrix}$

$$\overleftrightarrow{T}^t \overleftrightarrow{A} \overleftrightarrow{T} = \overleftrightarrow{T}^t \overleftrightarrow{M} \overleftrightarrow{T} \begin{bmatrix} k_1^*/m_1^* & 0 \\ 0 & k_2^*/m_2^* \end{bmatrix}$$

$$\left(\omega_i \equiv k_i^*/m_i^*\right) \longrightarrow = \overleftrightarrow{T}^t \overleftrightarrow{M} \overleftrightarrow{T} \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}$$

Multiply by $(\overleftrightarrow{T}^t)^{-1}$ on the left

$$\overleftrightarrow{A} \overleftrightarrow{T} = \overleftrightarrow{M} \overleftrightarrow{T} \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}$$

At this point, a component representation may be useful.

$$\sum_k A_{ik} T_{kj} = \sum_{k,l} M_{ik} T_{kl}\, \omega_l^2\, \delta_{lj} = \omega_j^2 \sum_k M_{ik} T_{kj}$$

$T_{kj}$ = j-th column vector of $\overleftrightarrow{T} \equiv \vec{T}_j$

Then, $\overleftrightarrow{A}\, \vec{T}_j = \omega_j^2\, \overleftrightarrow{M}\, \vec{T}_j$ $\qquad \overleftrightarrow{T} = [\underbrace{\vec{T}_1 \ \vec{T}_2}_{\text{column vectors}}]$

So, we arrive at an important form of our big Q:

$$\boxed{\text{Can we find } \vec{T}_j\text{'s such that } \quad \overleftrightarrow{A}\, \vec{T}_j = \omega_j^2\, \overleftrightarrow{M}\, \vec{T}_j \ ??}$$

The answer is a resounding yes for any (real) symmetric matrix $\overleftrightarrow{A}$ and any (real) symmetric positive definite matrix $\overleftrightarrow{M}$. (See, e.g., the "Numerical Recipe" book.)

In fact

$$\boxed{\overleftrightarrow{A}\,\vec{T}_j = \omega_j^2\,\overleftrightarrow{M}\,\vec{T}_j}$$

is called a generalized eigenvalue equation.

For symmetric $\overleftrightarrow{A}$, $\overleftrightarrow{M}$ and a positive definite $\overleftrightarrow{M}$, the solutions $\omega_j^2$ and $\vec{T}_j$ can always be found !!

Normal ~~modes~~ coordinates $\Rightarrow$ $\boxed{\vec{\eta} = \overleftrightarrow{T}^{-1}\,\vec{x}}$

Valid for any dimensions !!

For any # of coupled oscillators as long as $\overleftrightarrow{A}, \overleftrightarrow{M}$ = symmetric

$\overleftrightarrow{M}$ = positive definite.

- skip if you like -



Question (Advanced; "orthogonality")

We started with

(i) $\overleftrightarrow{T}^t \overleftrightarrow{M} \overleftrightarrow{T} = \begin{bmatrix} m_1^* & 0 \\ 0 & m_2^* \end{bmatrix}$ $\quad \overleftrightarrow{T}^t \overleftrightarrow{A} \overleftrightarrow{T} = \begin{bmatrix} k_1^* & 0 \\ 0 & k_2^* \end{bmatrix}$

and derived

(ii) $\overleftrightarrow{A} \overleftrightarrow{T} = \overleftrightarrow{M} \overleftrightarrow{T} \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix}$ $\quad \omega_i^2 \equiv k_i^* / m_i^*$

Can we ensure that (ii) implies (i)?

Yes! ① First, note $\vec{T}_i^t \overleftrightarrow{M} \vec{T}_i$ = positive due to the positive definite nature of $\overleftrightarrow{M}$.

②
$$\overleftrightarrow{A} \vec{T}_i = \omega_i^2 \overleftrightarrow{M} \vec{T}_i$$
$$\vec{T}_i^t \overleftrightarrow{A} = \omega_i^2 \vec{T}_i^t \overleftrightarrow{M}$$
$$\therefore \vec{T}_j^t \overleftrightarrow{A} \vec{T}_i = \omega_i^2 \vec{T}_j^t \overleftrightarrow{M} \vec{T}_i = \omega_j^2 \vec{T}_j^t \overleftrightarrow{M} \vec{T}_i$$
$$\therefore \vec{T}_j^t \overleftrightarrow{M} \vec{T}_i = 0 \text{ if } i \neq j$$

Combining ①, ② $\Rightarrow \overleftrightarrow{T}^t \overleftrightarrow{M} \overleftrightarrow{T} = \begin{bmatrix} m_1^* & 0 \\ 0 & m_2^* \end{bmatrix}$

$$m_1^*, m_2^* > 0$$

Then $\overleftrightarrow{T}^t \overleftrightarrow{A} \overleftrightarrow{T} = \overleftrightarrow{T}^t \overleftrightarrow{M} \overleftrightarrow{T} \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} = \begin{bmatrix} m_1^* \omega_1^2 & 0 \\ 0 & m_2^* \omega_2^2 \end{bmatrix}$

QED.

Consider our original problem

$$\overleftrightarrow{M} = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \qquad \overleftrightarrow{A} = \begin{bmatrix} k+K & -K \\ -K & k+K \end{bmatrix}$$

The eigenvalue problem

$$\overleftrightarrow{A}\,\vec{T}_i = \omega_i^2\,\overleftrightarrow{M}\,\vec{T}_i$$

$$\left(\overleftrightarrow{A} - \omega_i^2\,\overleftrightarrow{M}\right)\vec{T}_i = 0$$

(as in the Newton method)

For a non-trivial $\vec{T}_i$,

$$\left|\overleftrightarrow{A} - \omega_i^2\,\overleftrightarrow{M}\right| = \begin{vmatrix} k+K-M\omega_i^2 & -K \\ -K & k+K-M\omega_i^2 \end{vmatrix} = 0$$

$$\Rightarrow \quad \omega_i^2 = \frac{k}{M} \quad \text{or} \quad \frac{k+2K}{M} \quad \text{(as before)}$$

$$\vec{T}_i = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad\qquad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

up to a multiplicative constant.

Note that $\vec{x} = \overleftrightarrow{T}\vec{\eta} = [\vec{T}_1\ \vec{T}_2]\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$

when $\eta_1$ is excited only

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \eta_1\,\vec{T}_1 = \begin{bmatrix} \eta_1 \\ \eta_1 \end{bmatrix}$$

in-phase

$x_1 = x_2$

when $\eta_2$ is excited only

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \eta_2\,\vec{T}_2 = \begin{bmatrix} \eta_2 \\ -\eta_2 \end{bmatrix}$$

out-of phase

$x_1 = -x_2$

as before

In general $\vec{x} = \begin{bmatrix} \eta_1 \\ \eta_1 \end{bmatrix} + \begin{bmatrix} \eta_2 \\ -\eta_2 \end{bmatrix}$

$$* \begin{cases} x_1 = \eta_1 + \eta_2 \\ x_2 = \eta_1 - \eta_2 \end{cases} \Big\} \; \vec{x} = \overleftrightarrow{T}\, \vec{\eta}$$

$$\Rightarrow \quad \eta_1 = \frac{1}{2}(x_1 + x_2), \qquad \eta_2 = \frac{1}{2}(x_1 - x_2)$$

} corresponds to $\vec{\eta} = \overleftrightarrow{T}^{-1} \vec{x}$

$$\eta_1 = A_1 \cos(\omega_1 t + \phi_1) \qquad \omega_1 = \sqrt{\frac{k}{M}}$$

$$\eta_2 = A_2 \cos(\omega_2 t + \phi_2) \qquad \omega_2 = \sqrt{\frac{k+2K}{M}}$$

$$\begin{pmatrix} x_1 = A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2) \\ x_2 = A_1 \cos(\omega_1 t + \phi_1) - A_2 \cos(\omega_2 t + \phi_2) \end{pmatrix}$$

$\Rightarrow$ The general solution.
4 constants ($A_1, \phi_1, A_2, \phi_2$) because of 2 degrees of freedom.
($2 \times 2 = 4$)

So, the general solution is a linear combination of the two independent normal modes.

$A_1, \phi_1, A_2, \phi_2$ fixed by $x_1, \dot{x}_1, x_2, \dot{x}_2$ at $t = 0$.

## Grand Summary

Start with $T = \frac{1}{2} \dot{\vec{x}}^t \overleftrightarrow{M} \dot{\vec{x}}$

$U = \frac{1}{2} \vec{x}^t \overleftrightarrow{A} \vec{x}$

( $\vec{x}$ can be $\vec{q}$ (generalized coord.) )

Solve $\overleftrightarrow{A} \vec{T}_i = \omega_i^2 \overleftrightarrow{M} \vec{T}_i$ eigenvalue eq.

Secular eq $\left| \overleftrightarrow{A} - \omega_i^2 \overleftrightarrow{M} \right| = 0$

gives eigenvalues $\omega_i^2$ ← normal mode frequency

Plugging $\omega_i^2$ back to the eigenvalue eq.
one gets eigenvectors $\vec{T}_i$

$\vec{T}_i$ gives the info on $\underbrace{\text{normal mode}}_{\text{the } i\text{-th}}$, i.e, how the $x_j$'s are related to each other, through

$\vec{x} = \vec{T}_i \eta_i$ if only $\eta_i$ is excited.

The normal coordinates are given as

$\vec{\eta} = \overleftrightarrow{T}^{-1} \vec{x}$ $\quad \overleftrightarrow{T} = [\vec{T}_1 \vec{T}_2 \cdots \vec{T}_n]$

$n \times n$ matrix
↑ degrees of freedom

In general,

$\vec{x} = \overleftrightarrow{T} \vec{\eta}$ $\quad \eta_i = D_i \cos(\omega_i t + \phi_i)$

↑ integration constants ↑

n independent SHOs !! →

$$L = \sum_i \left( \frac{1}{2} \dot{\eta}_i^2 m_i^* - \frac{1}{2} \eta_i^2 m_i^* \omega_i^2 \right)$$