Notes for Lecture 13

Kepler problem

In the previous lecture, we set the scene for Kepler problem, by investigating the central force problem. Let us consider the two important cases: $U(r) = kr^2/2$ (Hooke's law) and $U(r) = -k/r$ (Kepler problem). As we learned in the last lecture, we have an effectively 2D problem with a conserved angular momentum. The nature of the motion in the Hooke's law case is easy. It is a 2D SHO (Section 3.3 of the book). As we demonstrated in class, and as shown in Figure 3.2 of the book, the orbits in this case is a circle, an ellipse, or a line, all with the period $\tau = 2\pi\sqrt{m/k}$ just like the associated 1D SHO that we know and love. The math in this case is very similar to what we did for 1D SHOs (pages 104,105 of the book).

In this lecture we discuss the Kepler problem.

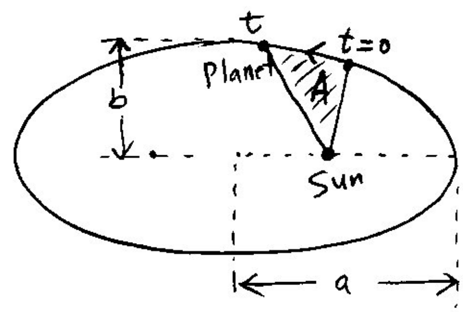
13.1 Kepler's laws

These laws apply to the motion of planets in our solar system. The diagram below is quite **exaggerated** in terms of the difference between a and b. As a matter of fact, most orbits of our planets are close to circles ($a \approx b$).

1. A planet's orbit is an ellipse with the Sun at one of the two focus points. See the diagram, which quite exaggerates the elliptical nature of the orbit. Most orbits are close to circles.

2. The areal velocity dA/dt is constant. See the diagram. The initial point ($t = 0$) can be arbitrarily chosen. A is the cumulative area swept by the position vector of the planet.

3. $\tau^2 \propto a^3$, where τ is the period of the motion, and a is the semi-major axis of the ellipse. a can be replaced by any linear dimension of the ellipse, such as the semi-minor axis b or the mean radius.



13.2 Kepler problem

These observational laws of Kepler can be proven if we use Newton's law of gravity for which

$$U(r) \ = \ -k/r$$

where $k = GM_S m_p$, M_s is the mass of the Sun, and m_p is the mass of a planet. Note that the reduced mass, relevant for the relative motion is given by $\mu = m_p M_S/(m_p + M_S) \approx m_p$ in the zero-th order as $M_S \gg m_p$. However, we will keep using μ, below, as this problem can describe any two body problem of celestial bodies. For instance, for a binary star consisting of two rotating equally massive stars, the reduced mass will be half of the individual mass.

The solution for the orbit can be found from the θ equation (the third equation) of page 6 of LN 12 ($\theta_0 \equiv 0$ and integrating after a change of variable $u = 1/r$): (see the derivation at the end of this LN)

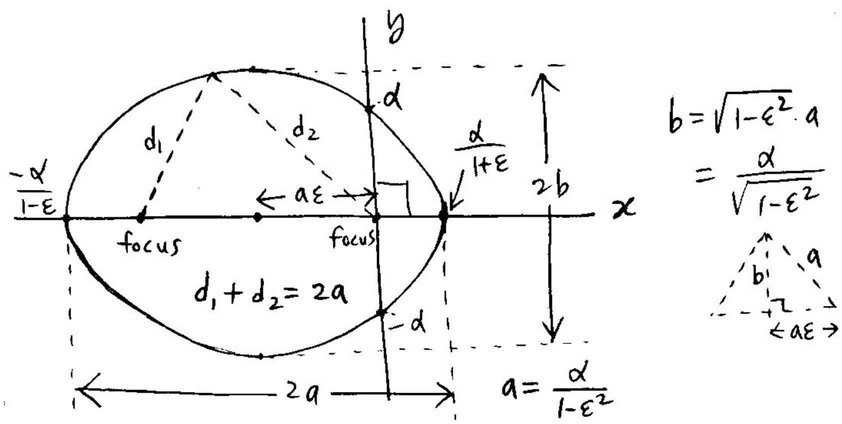
$$\frac{\alpha}{r} \ = \ 1 + \varepsilon \cos\theta$$

where $\alpha = l^2/\mu k, \ \varepsilon = \sqrt{1 + 2El^2/\mu k^2}$.

The shape of the orbit is the so-called "conic section" and it depends on the **eccentricity** ε. 2α is called the **latus lectum**.

Orbit	Eccentricity (ε)	Energy (E)	Note
Circle	0	$E = U_{eff,min}$	The easiest, the most important!
Ellipse	$0 < \varepsilon < 1$	$U_{eff,min} < E < 0$	
Parabola	1	$E = 0$	Escape condition, open orbit
Hyperbola	$\varepsilon > 1$	$E > 0$	Open orbit

Here is a summary of the geometry for the elliptical orbit and the circular orbit (**a special case when** $\varepsilon = 0$ **and** $a = b$):



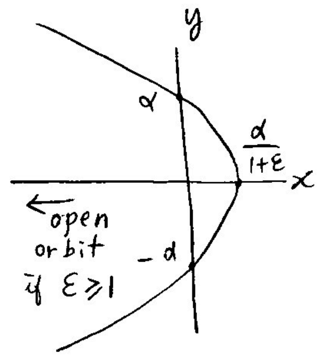
This proves Kepler's first law. The **semi-major axis** a and the **semi-minor axis** b are given by

$$a = \frac{\alpha}{1-\varepsilon^2} = \frac{k}{2|E|}$$

$$b = \frac{\alpha}{\sqrt{1-\varepsilon^2}} = \frac{l}{\sqrt{2\mu|E|}}$$

In the diagram, the fact that $a = \alpha/(1-\varepsilon^2)$ follows from the fact that $2a = \alpha/(1+\varepsilon) + \alpha/(1-\varepsilon)$. The fact the center of the ellipse lies at $x = -a\varepsilon$ follows from the fact that that x value must be the mean of $\alpha/(1+\varepsilon)$ and $-\alpha/(1-\varepsilon)$ (min/max of r, or apsides, corresponding to $\theta = 0$ and π). Then, $b = \alpha/\sqrt{1-\varepsilon^2}$ follows from the right triangle shown on the right side of the above diagram, considering the case $d_1 = d_2 = a$.

Before we discuss more about the elliptical orbits, here is a rough diagram for the parabolic or elliptical orbit. It can be considered as an elliptical orbit that becomes too large and eventually open up at the $x \to -\infty$ end as $\varepsilon \to 1$ (parabola) and greater (hyperbola).



[For the mathematically curious only...] The actual mathematical "proof" that the above equation $\frac{\alpha}{r} = 1 + \varepsilon\cos\theta$ really leads to all the conic section curves listed above, depending on the value of the eccentricity, is given now. Here, by "proof," I mean rendering the above polar coordinate equation to the "standard" Cartesian coordinate equation. Note that this equation can be written as $\alpha = r + r\varepsilon\cos\theta$ or $\alpha - \varepsilon x = r$. Squaring both sides, we get $(\alpha - \varepsilon x)^2 = x^2 + y^2$. It is immediately obvious that $\varepsilon = 1$ will render this equation to $\alpha^2 - 2\alpha\varepsilon x = y^2$, which represents a parabola. It is also trivial to see that $\varepsilon = 0$ means $r = \alpha$ (circle). It takes simple algebra to prove that $0 \le \varepsilon < 1$ will render the above equation to the form

$$1 = \frac{\left(x + \frac{\alpha\varepsilon}{1-\varepsilon^2}\right)^2}{a^2} + \frac{y^2}{b^2}$$

Here, a and b are as given above. This fits the general equation of an ellipse in the Cartesian coordinate system $(x - x_0)^2/a^2 + (y - y_0)^2/b^2 = 1$. The circle can be considered as a special ellipse, with $\varepsilon = 0$ leading to $a = b$. For $\varepsilon > 1$, we get

$$1 = \frac{\left(x - \frac{\alpha\varepsilon}{\varepsilon^2-1}\right)^2}{\left(\frac{\alpha}{\varepsilon^2-1}\right)^2} - \frac{y^2}{\left(\frac{\alpha^2}{\varepsilon^2-1}\right)}$$

This fits the general equation of a hyperbola in the Cartesian coordinate system $(x - x_0)^2/a^2 - (y - y_0)^2/b^2 = 1$.

[Back to the normal discussion ...] Let us come back to the discussion of elliptical orbits, applicable to comets as well as planets. In an elliptical orbit,

$$r_{min} = \frac{\alpha}{1+\varepsilon}, \; r_{max} = \frac{\alpha}{1-\varepsilon}$$

These are "turning points" for the r motion, or **apsides**. In particular, these are called **pericenter** (r_{min}), or **apocenter** (r_{max}). Think "apex" to remember which is which. For objects orbiting around the Earth, we talk about perigee and apogee, and for objects orbiting around the Sun, we talk about perihelion and aphelion. Note that **the speed is maximum (minimum) at r_{min} (r_{max})**. This follows directly from the energy conservation $E = T + U = \frac{1}{2}\mu v^2 - k/r$. Call them v_{min} and v_{max}. At the turning points, $\vec{v} \perp \vec{r}$, and so the angular momentum conservation means for apsides,

$$r_{min} v_{max} = r_{max} v_{min}$$

The total area of an ellipse is πab. As we saw, the areal velocity is constant **for any central force**, and is given by $dA/dt = l/2\mu$. This proves Kepler's second law (this is no news to us; we did this in the previous lecture). Now, the period τ for an elliptical motion is given by $\tau = \pi ab/(dA/dt) = 2\pi ab\mu/l$. As $b = \sqrt{\alpha}\sqrt{a} = l\sqrt{a}/\sqrt{\mu k}$, **we get the following Kepler's third law**

$$\tau^2 = \frac{4\pi^2\mu}{k} a^3$$

Note that this law should read generally thus: "the period squared is proportional to the linear dimension of the orbit cubed." For instance, if we had eliminated a instead of b, then we would have gotten $\tau^2 \propto b^3$ with a different proportionality constant. Or, one could say that the average radius of an elliptical orbit is proportional to $\tau^{2/3}$. However, as it is written, the above form, expressed in terms of a, has a certain beauty. Likewise, note that $E = -k/(2a)$ for an elliptical orbit (page 3, just below the ellipse diagram). This also has a certain beauty to it. The discussion in the following box will show why.

How to remember, or quickly re-derive, stuff?

Suppose you need to remember the above Kepler's third law (and some other crucial stuff that we derived above). **You can do it, very easily!** Do you have to memorize it? There is no way *I* can memorize such a formula! But, wait, there is a clever way to "recall," without doing all the complicated stuff that we just did! Just remember this one first: **the uniform circular motion is your best friend...** For the circular motion, we have

$$\frac{\mu v^2}{r} = \frac{k}{r^2} \qquad \text{Centripetal force equation}$$

$$E = T + U = \frac{\mu v^2}{2} - \frac{k}{r} \qquad \text{Energy conservation}$$

where r, v, T, U are all *constants*. By expressing the centripetal force equation in terms of τ, by noting $v = r\omega = r2\pi/\tau$, we get

$$\tau^2 = \frac{4\pi^2\mu}{k} r^3 \qquad \textbf{Kepler's 3rd law for circular orbit}$$

By multiplying the first equation by r, we get

$$-2T = U \qquad \textbf{Virial theorem example}$$

$$E = -T = \frac{U}{2} = -\frac{k}{2r}$$

The good news is that the last three equations remain valid even for elliptical orbits, if we make the following substitutions: ($\langle \cdots \rangle_\tau$: average over a period)

$$r \to a,\ T \to \langle T \rangle_\tau,\ U \to \langle U \rangle_\tau$$

For both circular orbits and elliptical orbits, we have:

$$\tau^2 = \frac{4\pi^2\mu}{k} a^3$$

$$\langle U \rangle_\tau = -2\langle T \rangle_\tau$$

$$E = -\langle T \rangle_\tau = \frac{\langle U \rangle_\tau}{2} = -\frac{k}{2a}$$

The angular momentum conservation, $l = \mu r^2 \dot{\theta} = 2\mu dA/dt$ (valid for *any* central force) and these three equations are **crucial things to rather easily recall** as shown here (not memorize!).

Required Reading

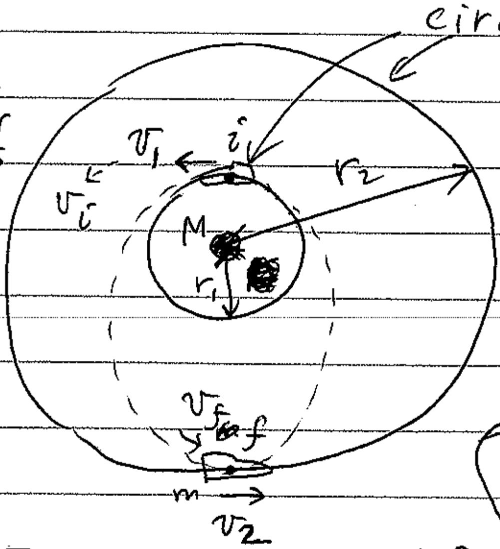
§ . Hohmann transfer

- The most fuel-efficient way (to go to a different circular orbit.)

("Shuttle orbit change paradox")

v_1, v_2 circular motions

v_i, v_f elliptical motion



Quick Forward thrusts at i, f.

Gravitational ~~force~~ $-\frac{k}{r^2}$, $k = GMm$

- Increase kinetic energy twice at i and f.
  $v_1 < v_i \rightarrow v_f < v_2$
- Yet, end up with $v_2 < v_1$.
- Solution ? $v_f < v_i$

① For circular motion, $r_1 v_1^2 = r_2 v_2^2 = \frac{k}{m}$ (from $2T = -U$)

② Kepler's 2nd law ($\vec{L}$ conservation) $\Rightarrow r_1 v_i = r_2 v_f$

③ Energy conservation $\Rightarrow \frac{1}{2} m v_i^2 - \underbrace{\frac{k}{r_1}}_{-m v_1^2} = \frac{1}{2} m v_f^2 - \underbrace{\frac{k}{r_2}}_{-m v_2^2}$

Two equations $r_1 v_i = r_2 v_f$

$$\boxed{v_i^2 - v_f^2 = 2(v_1^2 - v_2^2)} \quad \begin{pmatrix} v_1 > v_2 \\ v_i > v_f \end{pmatrix}$$

$$v_i = \sqrt{\frac{2 r_2^2}{r_2^2 - r_1^2}} \cdot \sqrt{v_1^2 - v_2^2} = \sqrt{\frac{2 r_2^2}{r_2^2 - r_1^2}} \cdot \sqrt{1 - \frac{r_1}{r_2}} \cdot v_1$$

$$v_f = \sqrt{\frac{2 r_1}{r_1 + r_2}} \cdot v_2 = \sqrt{\frac{2 r_2}{r_1 + r_2}} \cdot v_1$$

④ Time of transfer $= \pi \sqrt{\frac{m}{GMm}} \cdot \left(\frac{r_1 + r_2}{2}\right)^{3/2} \Rightarrow$ (~259 days from E to Mars)

Optional Reading "Just Math"

Derivation of

$$\boxed{\frac{\alpha}{r} = 1 + \varepsilon \cos\theta}$$

$$\theta = \int_{r_0}^{r} dr' \cdot \frac{\ell}{r'^2 \sqrt{2\mu\left(E - U(r')\right) - \frac{\ell^2}{r'^2}}} \Rightarrow$$

$U(r') \to -\frac{k}{r'}$

$$\alpha = \frac{\ell^2}{\mu k}, \quad \varepsilon = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}$$

$$\frac{1}{r'} = u, \quad du = -\frac{dr'}{r'^2}$$

$$\theta = -\int_{u_0}^{u} du \cdot \frac{\ell}{\sqrt{2\mu(E + ku) - \ell^2 u^2}}$$

$$\to \sqrt{2\mu E + \frac{\mu^2 k^2}{\ell^2} - \ell^2\left(u - \frac{\mu k}{\ell^2}\right)^2}$$

$$= \ell \sqrt{\frac{2\mu E}{\ell^2} + \frac{\mu^2 k^2}{\ell^4} - \left(u - \frac{\mu k}{\ell^2}\right)^2} \qquad \frac{1}{\alpha} \equiv \frac{\mu k}{\ell^2}$$

$$= \ell \sqrt{\left(\varepsilon^2\right)\frac{1}{\alpha^2} - \left(u - \frac{1}{\alpha}\right)^2} \qquad \varepsilon \equiv \sqrt{1 + \frac{2E\ell^2}{\mu \ell^2}}$$

$$\theta = -\int_{u_0}^{u} du \cdot \frac{1}{\sqrt{\frac{\varepsilon^2}{\alpha^2} - \left(u - \frac{1}{\alpha}\right)^2}}$$

$$u - \frac{1}{\alpha} = \frac{\varepsilon}{\alpha} \cos\xi$$

$$\theta = \cos^{-1}\xi \Rightarrow \xi = \theta$$

$$\frac{1}{r} - \frac{1}{\alpha} = \frac{\varepsilon}{\alpha}\cos\theta \Rightarrow \boxed{\frac{\alpha}{r} = 1 + \cos\theta}$$