§. Lagrangian EOM with constraint(s)

Let us consider one constraint only.
(generalization to multiple constraints straight-f.)

(★) Assume $\underbrace{\sum_k a_k \dot{q}_k + a_t = 0}_{\text{a constraint } \sim\sim\text{that satisfies}\sim\sim}$.

a_k, a_t → functions of q_k, t

In general, this cannot be integrated ("non-holonomic")
If it can be integrated, then we have a holonomic constraint.
(e.g, $v = R\omega$ for a rolling w/o slipping)

The above equation can be written as

$$\sum_k a_k \, dq_k + a_t \, dt = 0$$

For a virtual displacement $\delta t = 0$

$$\sum_k a_k \, \delta q_k = 0 \quad \cdots (★)$$

$$\int_1^2 dt \sum_k \left(\frac{\partial L}{\partial q_k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k = 0$$

Add $\underset{(★)}{0}$ times $\lambda(t)$ --- Lagrange undetermined multiplier

$$\int_1^2 dt \sum_k \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} + \lambda(t) a_k \right] \delta q_k = 0$$

Not all q_k's are independent!

A special commonplace sub-class of all possible constraints $F(q_k, \dot{q}_k, t) = 0$ (all k's $1 \cdots n$)

Holonomic constraint: $F(q_k, \dot{q}_k, t) = G(q_k, t) = 0$

simple. Can reduce the # of the generalized coordinates by 1 per constraint.

But only one constraint.

From $k = 1, \ldots, n$, if we ~~remove~~ exclude
(n = D.O.F.)

one of q_k's, then the rest are independent!

So, consider $q_2, \ldots, q_n$ as indep. vars.

As for q_1, choose $\lambda(t)$ so that

$$\frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} + \lambda(t)\, a_1 = 0 .$$

As for other q_k's, ($k = 2, \ldots, n$) the variational principle as before

$$\Rightarrow \frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} + \lambda(t)\, a_k = 0$$

Result? This equation is valid for all k's including $k=1$.

★

Price? $n+1$ unknowns $q_k, \lambda(t)$ $\quad k=1, \ldots, n$

Only n Lagrange eq's ?! No worries. We have the constraint equation

$$\sum_k a_k \dot{q}_k + a_t = 0$$

Why write in this form??

See L9-7!

$-\frac{\partial L}{\partial q_j} + \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = Q_j$ = generalized force

Meaning of $\lambda(t)$: generalized force of constraint

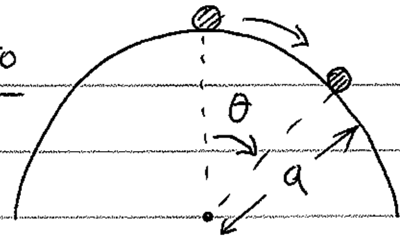
$$-\frac{\partial L}{\partial q_1} + \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} = \boxed{+\lambda(t)\, a_1}$$

★

Lagrangian mechanics is cool/sneaky in this regard. No need to know the constraint force to start with. It is an outcome! (friction ... normal force ...)

$\theta = \dot{\theta} = 0$ at $t=0$

☆ Ex. 7.10



r, θ

$$T = \frac{1}{2} m\left\{\dot{r}^2 + (r\dot{\theta})^2\right\}$$

$$U = mgr\cos\theta$$

$$L = \frac{1}{2} m\left\{\dot{r}^2 + (r\dot{\theta})^2\right\} - mgr\cos\theta$$

① $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{r}} - \dfrac{\partial L}{\partial r} = \lambda \dfrac{\partial f}{\partial r}$ 1 $f(r,\theta) = r - a = 0$

② $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{\theta}} - \dfrac{\partial L}{\partial \theta} = \lambda \dfrac{\partial f}{\partial \theta}$ 0 $\dfrac{\partial f}{\partial r} = 1$, $\dfrac{\partial f}{\partial \theta} = 0$

① $m\ddot{r} - mr\dot{\theta}^2 + mg\cos\theta = \lambda$

② $\dfrac{d}{dt}(mr^2\dot{\theta}) = mgr\sin\theta$

③ $r = a$ ② $\Rightarrow$ $\ddot{\theta} = \dfrac{g}{r} s_\theta$ $\dot{\theta}\dfrac{d\dot{\theta}}{d\theta}$ $\dfrac{\dot{\theta}^2}{2}$ E const.

① $\Rightarrow$ $-mr\dot{\theta}^2 + mg\cos\theta = \lambda$

④ $T + U = E = \dfrac{1}{2} m\left\{\dot{r}^2 + (r\dot{\theta})^2\right\} + mgr\cos\theta = mga$

$$\boxed{\frac{1}{2} a\dot{\theta}^2 + g\cancel{a}c_\theta = g\cancel{a}}$$

λ : normal force

$(3c_\theta - 2)mg$

$\lambda = 0$ ... $a\dot{\theta}^2 = g c_\theta$

$\lambda > 0$ ... ok.

$\lambda < 0$ ... ~~not~~ ok. $\dfrac{3}{2} g c_\theta = g$ $\therefore$ $c_\theta = \dfrac{2}{3}$

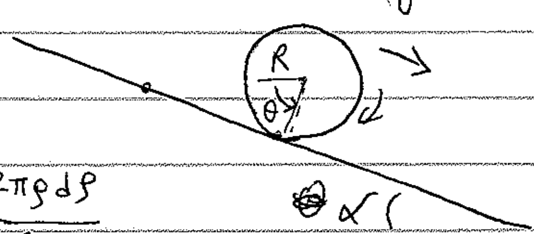
The particle takes off from the surface. $\leftarrow$ $\boxed{\theta = \cos^{-1}\dfrac{2}{3}}$

Was that easier than doing $\left(\begin{array}{l} mg\cos\theta = m\dfrac{v^2}{r} \\ \dfrac{1}{2}mv^2 + mga\cos\theta = mga \end{array}\right)$?

No!

do it

Ex 7.9



$$M\int_0^R \rho^2 \frac{2\pi\rho\, d\rho}{\pi R^2}$$

Constraint $v = R\dot{\theta}$

$s = R\theta$

s: distance traveled from rest

$U = -Mgs\sin\alpha$

$v = \dot{s}$

$ds - R\,d\theta = 0$

$$T = \frac{1}{2}Mv^2 + \frac{1}{2}\cdot\frac{1}{2}MR^2\cdot\dot{\theta}^2$$

$$= \frac{1}{2}M\dot{s}^2 + \frac{1}{4}MR^2\dot{\theta}^2$$

$$L = \frac{1}{2}M\dot{s}^2 + \frac{1}{4}MR^2\dot{\theta}^2 + Mgs\sin\alpha$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = -\lambda(t)R$$

$\dot{s} = R\dot{\theta}$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{s}} - \frac{\partial L}{\partial s} = \lambda(t)$$

$$\frac{1}{2}MR^2\ddot{\theta} = -\lambda(t)R$$

$$M\ddot{s} - Mg\sin\alpha = \lambda(t) = -\frac{1}{2}MR\ddot{\theta}$$

$MR\ddot{\theta}$

$$\boxed{\ddot{\theta} = \frac{2\sin\alpha}{3R}g}$$

→ 0 as $\alpha \to 0$

Non-sense!!

This model is highly idealized. Not enough for real rolling wheels.

$$\lambda(t) = -\frac{1}{3}Mg\sin\alpha \quad \leftarrow \text{friction}$$

$$-\lambda(t)R = \frac{1}{3}MgR\sin\alpha \quad \leftarrow \text{torque}$$

Some ending comments (Advanced)
(Reading Material.)

(1) $L \Rightarrow \alpha L$: doesn't change a thing.

"Mechanical similarity"

For example, $L = \frac{1}{2} m v^2 - m g y$ — projectile motion

Scale space $y, x \to Ay, Ax$

Scale time $t \to \sqrt{A}\, t$

$v \to \sqrt{A}\, v$

$$L \Rightarrow A L$$

$y \propto t^2$

$y \propto v^2$ → $h \propto v_0^2$ when thrown upwards

$$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

$x \to Ax$

$t \to t$

$L \Rightarrow A^2 L$

Period is independent of amplitude!

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{\alpha}{r}$$ Kepler prob.

$x, y, z \to Ax, Ay, Az$

$t \to A^{3/2}\, t$

$L \Rightarrow \frac{1}{A} L$

Related to the Lagrangian formulation in a non-inertial frame

$T \propto L^{3/2}$

T : period

L : orbit dimension

: Kepler's third law!

(2) $L \Rightarrow L + \frac{df}{dt}$: doesn't change a thing.

$f = f(q, \cancel{\dot{q}}, t)$