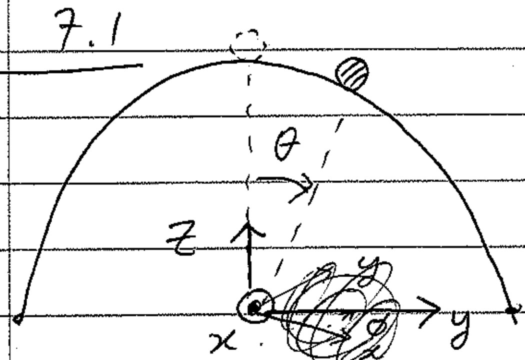
* Ex. 7.1 ↗ Do it w/ Ex 7.10



$$x^2+y^2+z^2=R^2$$
$$z \geq 0$$

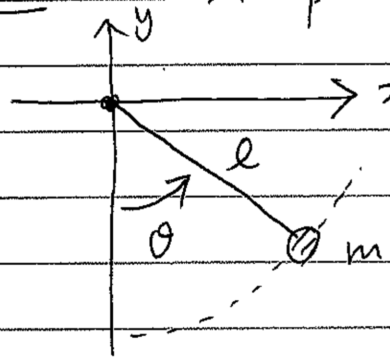
Generalized coordinates?

x, y ok

θ, ϕ better?

Do it

* Ex 7.2 Simple pendulum



$$T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2$$

no radial motion. only tangential

$$U = -mgl\cos\theta$$

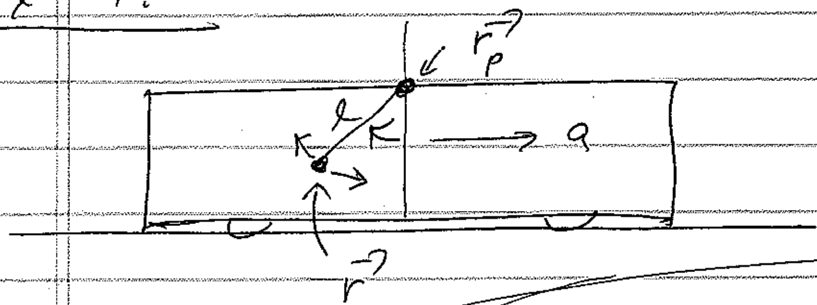
$$L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$$

E-L eq.

$$\underbrace{ml^2\ddot{\theta}}_{I\alpha\ (\Leftarrow \frac{dL}{dt})} = \underbrace{-mgl\sin\theta}_{\text{torque}}$$

Explit t-dependence in L.

Ex 7.6



OK to ignore it to start with. But will keep it just to show that it does not matter.

$$\vec r_p = (\cancel{v_0 t} + \tfrac{1}{2} a t^2)\,\hat x$$

$$\vec r = (\cancel{v_0 t} + \tfrac{1}{2} a t^2 + l \cancel{\cos\theta} \sin\theta)\,\hat x - l\cos\theta\,\hat y$$

$$\vec v = (v_0 + at + l\dot\theta\cos\theta)\,\hat x + l\dot\theta\sin\theta\,\hat y$$

$$T = \tfrac{1}{2} m\left[(v_0 + at + l\dot\theta c_\theta)^2 + (l\dot\theta s_\theta)^2\right]$$

$$U = -mgl\,c_\theta$$

$$L = \tfrac{1}{2} m\left[(v_0 + at + l\dot\theta c_\theta)^2 + (l\dot\theta s_\theta)^2\right] + mgl\,c_\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = \frac{d}{dt}\left\{m\left[(v_0 + at + l\dot\theta c_\theta)\,l c_\theta + l\dot\theta s_\theta \cdot l s_\theta\right]\right\}$$

$$\frac{\partial L}{\partial \theta} = -m(v_0 + at)\,l\dot\theta\,s_\theta - mgl\,s_\theta$$

$$\to \frac{d}{dt}\left[m\left\{(v_0 + at)\,l c_\theta + l^2\dot\theta\right\}\right]$$

$$= m\left(a l c_\theta - (v_0 + at)\,l s_\theta\,\dot\theta + l^2\ddot\theta\right)$$

$$l^2\ddot\theta = -\cancel{m}gl\,s_\theta - \cancel{m}al\,c_\theta \qquad \tan\theta_E = \frac{-a}{g}$$

$$\theta \Rightarrow \phi + \theta_E$$

$\theta = \theta_\varepsilon + \phi$ $\quad\quad \phi$ = small angle

(in class $\Rightarrow \theta_{eq} + \delta$)

$$\ell \ddot{\phi} = -g\, s_{\phi+\theta_\varepsilon} - a\, c_{\phi+\theta_\varepsilon} \qquad t_{\theta_\varepsilon} = -\frac{a}{g}$$

$$s_\phi c_{\theta_\varepsilon} + \cancel{c_\phi} s_{\theta_\varepsilon} \qquad \cancel{c_\phi} c_{\theta_\varepsilon} - s_\phi s_{\theta_\varepsilon}$$

$$\simeq -g\, c_{\theta_\varepsilon} \phi + a\, s_{\theta_\varepsilon} \phi \qquad c_{\theta_\varepsilon} = \frac{g}{\sqrt{a^2+g^2}}$$

$$\ell \ddot{\phi} = -\sqrt{g^2+a^2}\ \phi \qquad s_{\theta_\varepsilon} = -\frac{a}{\sqrt{a^2+g^2}}$$

SHO!

$$\omega_0^2 = \frac{\sqrt{g^2+a^2}}{\ell}$$

Notes for Lecture 10

Symmetry, conservations, and H

Conservation principles are very fundamental in physics. Most conservation principles that we know arise from symmetry.

10.1 Symmetry principles

Symmetry is important. It really is. A well-known Nobel laureate theorist once said that about three quarters of all physics is about the symmetry.[1]

The symmetry may sound difficult. It really isn't.

Symmetry can be discussed in such grand language as "the homogeneity of time and space" and "the isotropy of space," and more. And, the symmetry can be discussed and analyzed at length by the wonderful mathematics that is the group theory.

But let us ask what it is really ... apart from all that grand or mathematical language.

The essential physics about symmetry is really simple.

Say, you do an experiment of some kind. On a table top. Perhaps you are colliding two balls. Or, you are making some chemical reaction (that is outside the realm of classical mechanics, but that is alright – symmetry is applicable in all physics). Let us assume that your experiment does not depend on any other thing than what is on

[1] To be more precise, what is important is how symmetry breaks (spontaneous symmetry breaking) as much as how symmetry leads to conserved quantities.

the table. It is not affected by the Earth's spinning, the Earth magnetic field, etc. Also, it does not depend on whether you turn on the room light or not. In other words, we can say your experiment on that table forms a "closed system." This is our assumption, which would be satisfied reasonably well by many kinds of experiments.

Now, suppose you do the experiment today, and you do it again tomorrow under identical conditions. You would expect that the experiment will give exactly the same result. Well, that may be slightly misleading. This is what I mean. Any experiment will be affected by statistical fluctuations and you can never obtain exactly the same result in the literal sense. However, when those fluctuations are averaged out, your result of today should agree with your result of tomorrow. Let us say that that is what we actually mean by "getting the same result." This is the symmetry that we call "**the homogeneity of time**."

It means that physical laws should not change over time. This is true as far as we know.

Now, you imagine doing the same experiment, but do it in one corner of the room and then do it in another corner of the room, after simply translating the table, that is after simply rolling the table with all stuff on it intact. Again, you would expect to get the same result in the new corner, as you did in the first corner. You would expect, in fact, that the experiment will work no matter where you put the table. This is the symmetry that we call "**the homogeneity of space**."

It means that physical laws should not change just because we simply moved from point 1 to point 2 in the Universe. This is true as far as we know.

Physicists attribute this particular symmetry to Newton, who had this legendary epiphany that maybe the apple falling from a tree works the same way as the Moon going round the Earth. What a breathless moment he must have had on such a moment of insight!

Finally, let us imagine that you do an experiment. And then you rotate the whole set up by some angle and then do the experiment again. This time too you would expect that you get the same result. This is the symmetry that we call "**the isotropy of space**."

These are some of the basic symmetries that we physicists have learned as fundamental principles of Nature.

There are other symmetries, of course. Many more, in fact. But we won't need them here in this course. Just one more example. The theory of relativity is another famous theory about symmetry – symmetry of moving reference frames. Indeed, Einstein is the one who gets credit for putting the symmetry principle at the forefront of physics, through his very successful theory of relativity.

So, the concept of symmetry is quite plain and simple. But let us mention one more thing. A very important thing. You probably know that science works only so far as experiments are reproducible. If the results of an experiment on a closed system depend on time and space, then it means that group A and group B who perform the same experiment can never compare their notes. [In such an experiment, one might suspect that the assumption of a "closed system" is incorrect.] Such a situation forbids scientific progress of any kind. The point here is this. The concept of symmetry is not an idle, abstract, or vacant, one at all. If these symmetries did not exist, then science as we know it would not exist.

10.2 Conservation principles

Another reason that "symmetry" is important is because it leads to conservation principles. **This is true in all physics.** Here, we consider the consequences in classical mechanics, of the three kinds of symmetry mentioned above.

Let us pause here one minute, though. Let us think – what does it mean when we say that "we should expect the same experimental result" when we do the experiment at a different time, at a different position, or at a different orientation in space? It means that, despite the difference in time, or the different in position, or the difference in orientation, if we prepare the experiment the same way, i.e. if we give the same initial conditions, then the final outcome must be the same. This can only be possible if the mechanical law is invariant under the corresponding "symmetry operations," translation in time or space or rotation in space. In other words, for a closed system, we require that Newton's laws are invariant on translation in time or space or rotation in space. But, as we all know, the Lagrangian is a much nicer scalar quantity from which Newton's law can be written down, so we might as well say the following.

> For a closed mechanical system, the Lagrangian is invariant on translation in time, on translation in space, or on rotation in space. A closed system means a system of particles that interact only within themselves. We attribute this to the homogeneity of time, the homogeneity of space, and the isotropy of space.

One should note if the Lagrangian is invariant then the equation of motion is invariant too, since the Lagrangian equation of motion, which is equivalent to the Newtonian equation of motion, is derived directly from the Lagrangian.

When the Lagrangian is invariant under a "symmetry operation" (such as translation in time or space, or rotation in space), a conserved quantity is guaranteed (energy, momentum, or angular momentum, respectively for these examples). Such conserved quantities should exist, then, trivially for closed systems. But this is not all. **Even for a non-closed system**, such symmetry can exist in full or in part. Then, conserved quantities follow, in full or in part. This property generally goes by the name of "Noether theorem."

10.2.1 Momentum conservation

Let us consider this "**translational symmetry in space**" first. Or, the symmetry reflecting the **homogeneity of space** as we discussed above.

Consider transforming the coordinate system such that each position vector is transformed as

$$\vec{r} \to \vec{r} + \delta\vec{r}$$

where $\delta\vec{r}$ is a constant, but arbitrary, vector. This means $\dot{\vec{r}}$ does not change. The Lagrangian $L = L(\dot{r}, \dot{\vec{r}}, t)$, would change accordingly

$$L(\vec{r}, \dot{\vec{r}}, t) \to L(\vec{r} + \delta\vec{r}, \dot{\vec{r}}, t) = L(\vec{r}, \dot{\vec{r}}, t) + \frac{\partial L}{\partial \vec{r}} \cdot \delta\vec{r}$$

where

$$\frac{\partial}{\partial \vec{r}} \stackrel{def}{=} \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

Now, we can apply the Lagrange equation

$$\frac{\partial L}{\partial \vec{r}} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{r}}}$$

where

$$\frac{\partial}{\partial \dot{\vec{r}}} \stackrel{def}{=} \hat{x}\frac{\partial}{\partial \dot{x}} + \hat{y}\frac{\partial}{\partial \dot{y}} + \hat{z}\frac{\partial}{\partial \dot{z}}$$

to conclude that

$$L(\vec{r}, \dot{\vec{r}}, t) \to L(\vec{r} + \delta\vec{r}, \dot{\vec{r}}, t) = L(\vec{r}, \dot{\vec{r}}, t) + \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{r}}}\right) \cdot \delta\vec{r}$$

Canonical momentum: for a given generalized coordinate q, the quantity

$$p \stackrel{def}{=} \partial L/\partial \dot{q}$$

defines the canonical momentum for q.

For a closed system

$$L(\vec{r}, \dot{\vec{r}}, t) = L(\vec{r} + \delta\vec{r}, \dot{\vec{r}}, t)$$

<u>L is "translationally invariant"</u>

$$= L + \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{r}}}\right)\cdot \delta\vec{r}$$

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{r}}}\right)\cdot \delta\vec{r} = 0 \quad \text{for any } \underset{\text{small}}{\wedge}\ \delta\vec{r}$$

By taking $\delta\vec{r} = \delta x\,\hat{x}$, $\delta y\,\hat{y}$ and $\delta z\,\hat{z}$ in turn, we conclude

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec{r}}} = \boxed{\frac{d}{dt}\vec{p} = 0}$$

$\vec{p}$: canonical momentum for $\vec{r}$ = linear momentum

Momentum conservation $\Leftarrow$

Even for a non-closed system

$$\left(\frac{d}{dt}\vec{p}\right)\cdot\delta\vec{r} = 0$$

can be valid for $\delta\vec{r}$ in <u>some</u> direction(s).

Then $\dfrac{d\vec{p}}{dt} = 0$ in <u>those</u> directions.

Ex) ① Free particle $L = \frac{1}{2}m\vec{v}^2 = \frac{1}{2}m\dot{\vec{r}}^2$

$$\Rightarrow \vec{p} = \frac{\partial L}{\partial \dot{\vec{r}}} = m\dot{\vec{r}}$$

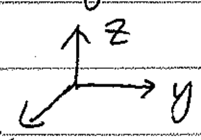
$\Downarrow$

translationally invariant (don't depend on $\vec{r}$ at all)

$\therefore$ <u>$\vec{p}$ conserved</u>

② Particle in a constant $\vec{g}$ field.

$$L = \frac{1}{2}m\dot{\vec{r}}^2 - mgz$$



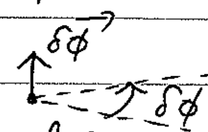
L is translationally invariant for x, y

$\underbrace{P_x = mv_x,\ P_y = mv_y}$ $\rightarrow$ conserved

§. Conservation of angular momentum

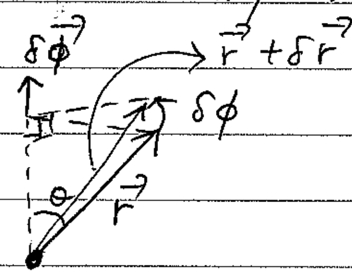
Consider a small rotation by $\delta\phi$ ← Constant but arbitrary

Define a vector $\delta\vec{\phi}$ as

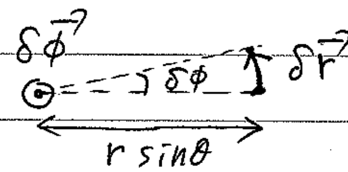


(right screw rule)

For an arbitrary vector $\vec{r}$



Top View



$\delta\vec{r} \perp \delta\vec{\phi},\ \vec{r}$ $\qquad |\delta\vec{r}| = r\sin\theta\,\delta\phi$

$$\delta\vec{r} = \delta\vec{\phi}\times\vec{r}$$

$$\delta\dot{\vec{r}} = \delta\vec{\phi}\times\dot{\vec{r}}$$

$$L(\vec{r},\dot{\vec{r}},t) \rightarrow L(\vec{r}+\delta\vec{\phi}\times\vec{r},\ \dot{\vec{r}}+\delta\vec{\phi}\times\dot{\vec{r}},\ t) = L+\delta L$$

$$= \underbrace{\frac{\partial L}{\partial\vec{r}}}_{=\frac{d}{dt}\vec{p}}\cdot(\delta\vec{\phi}\times\vec{r}) + \underbrace{\frac{\partial L}{\partial\dot{\vec{r}}}}_{=\vec{p}}\cdot(\delta\vec{\phi}\times\dot{\vec{r}})$$

$\delta L = 0$

L is "rotationally invariant"

$$= \delta\vec{\phi}\cdot\left(\vec{r}\times\frac{d\vec{p}}{dt}\right) + \delta\vec{\phi}\cdot(\dot{\vec{r}}\times\vec{p})$$

$$= \delta\vec{\phi}\cdot\frac{d}{dt}(\vec{r}\times\vec{p}) = \delta\vec{\phi}\cdot\frac{d\vec{L}}{dt}$$

For a closed system, for $\delta\vec{\phi}$ arbitrary in all directions

$\therefore\ \dfrac{d\vec{L}}{dt} = 0 \qquad \vec{L}$ is conserved.

For an open system, L can be invariant for certain directions of $\delta\vec{\phi}$

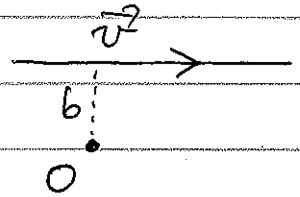
$\Rightarrow$ $\vec{L}$ in <u>those</u> directions are conserved.

<u>Note</u>. $\vec{L}$ <u>depends on</u> the origin.

Ex) ① Free particle $\quad L = \frac{1}{2} m \dot{\vec{r}}^2$

Rotationally invariant

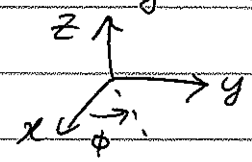
$\Rightarrow$ $\vec{L}$ is a constant.



$\Rightarrow$ $\vec{L} = mvb \otimes$ direction

② Particle in a constant $\vec{g}$ field.

$$L = \frac{1}{2} m \dot{\vec{r}}^2 - mgz$$



L is invariant for $\delta\vec{\phi} \parallel \hat{z}$.

$\Rightarrow$ L_z is conserved.

v_x, v_y constant

$v = \sqrt{v_x^2 + v_y^2}$

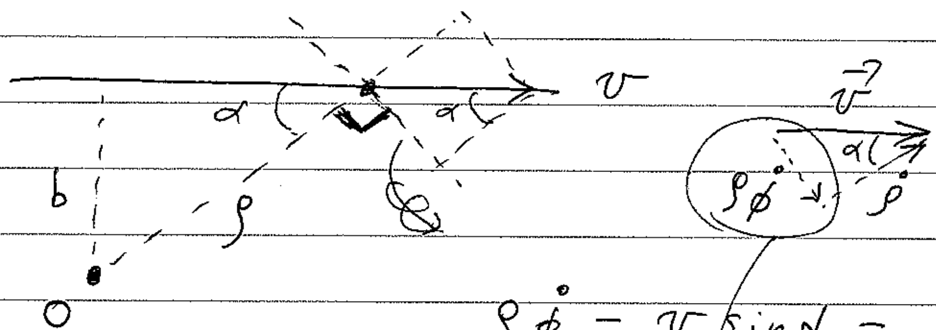
<u>Note</u>) Rotational invariance = translational invariance for an angle coordinate

$q = \phi \qquad \underset{\text{canonical momentum}}{p} = L_\phi$ = angular momentum $\rightarrow \rho^2\dot{\phi}^2$

different L's!

② $\rightarrow L = \frac{1}{2} m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) - mgz$ $\Leftarrow$ in cylindrical coord.

$\rightarrow L_\phi = \frac{\partial L}{\partial \dot{\phi}} = m\rho^2\dot{\phi}$ = "$I\omega$" = angular momentum



$$\rho\dot{\phi} = v\sin\alpha = v\cdot\frac{b}{\rho}$$

$$L_\phi = m\rho^2\dot{\phi} = mvb$$

direction $\rightarrow \otimes$

§. Hamiltonian Mechanics

$$dL = \underbrace{\overbrace{\frac{\partial L}{\partial q}}^{\dot{p}} dq + \overbrace{\frac{\partial L}{\partial \dot{q}}}^{p} d\dot{q} + \frac{\partial L}{\partial t}\overbrace{dt}^{dt}}$$

Consider $H = p\dot{q} - L$ (Legendre transformation)

$$dH = dp\,\dot{q} + p\,d\dot{q} - dL$$

$$= \dot{q}\,dp - \dot{p}\,dq - \frac{\partial L}{\partial t}dt$$

[indep. variables $q, \dot{q}, t \Rightarrow q, p, t$]

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

$$\boxed{\frac{\partial H}{\partial p} = \dot{q}, \quad \frac{\partial H}{\partial q} = -\dot{p}}$$

Canonical equations of motion

$$\boxed{\frac{dH}{dt} = \dot{q}\dot{p} - \dot{p}\dot{q} - \frac{\partial L}{\partial t} = -\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}}$$

For many degrees of freedom

$$\boxed{H = \sum_i p_i\dot{q}_i - L} \qquad \boxed{\frac{\partial H}{\partial p_i} = \dot{q}_i, \quad \frac{\partial H}{\partial q_i} = -\dot{p}_i}$$

§. Principle of (energy or H) conservation.

$$\frac{dH}{dt} = 0 \quad \text{if } L \text{ (or } H\text{) does not depend on } t \text{ explicitly.}$$

(homogenity of time)

H : hamiltonian

proof

$$\frac{dH}{dt} \underset{\text{prev. page}}{=} \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

It is the "energy" in most, but not all, cases.

If $L = \frac{1}{2} m \vec{v}^2 - U(\vec{r})$

$$\frac{\partial L}{\partial \dot{\vec{r}}} = m\vec{v} = \vec{p}$$

$$H = \vec{p} \cdot \vec{v} - \frac{1}{2} m \vec{v}^2 + U(\vec{r})$$

$$= \vec{p} \cdot \frac{\vec{p}}{m} - \frac{\vec{p}^2}{2m} + U(\vec{r})$$

$$= \frac{\vec{p}^2}{2m} + U(\vec{r})$$

$$= T + V$$

Note : H is a function of p, q
not of $\dot{q}, q$

$$L(q, \dot{q}, t) \longrightarrow H(p, q, t)$$

$$\searrow p\dot{q} - L \nearrow$$

§. Some words about H and other aspects of the Lagrangian Mechanics

① The canonical EoM $\dot{p} = -\frac{\partial H}{\partial q}$, $\dot{q} = \frac{\partial H}{\partial p}$

are not used often to do problems. (Lagrangian does just fine.)
Rather, the Hamiltonian dynamics is important as a formalism that connects to one of the common formalism of QM --- the (Heisenberg / Schrödinger) formalism. One can say that if one quatizes the Poisson bracket (see homework) then the QM results.

② H is important in the sense that it describes mechanics in the "true phase space," (p, q).
p and q are canonically "conjugate variables".
For instance, the Louiville's theorem holds in the (true) phase space. (Read 7.12) The essence of this theorem is that if you prepare an "ensemble" of many systems which have different initial conditions then the density of the points that represent those systems in the phase space is constant.
(Important in charged particle optics, cosmology, etc..)

③ Virial theorem (7.13)
For power-law force, pot. $U \propto r^n$

$$2\langle T\rangle = n\langle U\rangle$$

average over a long time or a period