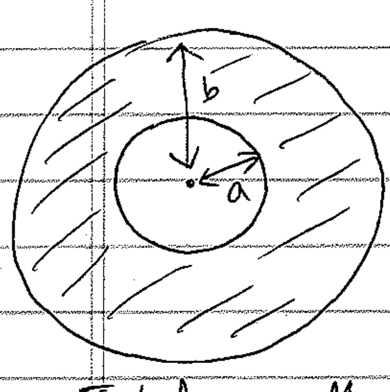
Ex 5.1 Spherically symmetric problem.
Gauss law problem!



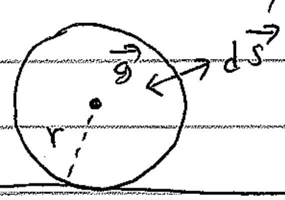
Total mass M

mass between $a < r < b$ only
mass "shell"

Due to the spherical symm.
$g = |\vec{g}|$, Φ depend only on r
not on θ, ϕ

(r, θ, ϕ) --- spherical coord. sys.

Gauss law at any r



$-g \cdot 4\pi r^2 =$ mass inside radius r $\times (-4\pi G)$

① If $r < a$, $g = 0$ ∵ no mass inside

② If $r > b$, $g = \frac{GM}{r^2}$ $\quad \vec{g} = -\frac{GM}{r^2}\hat{r}$

Just as though there is a point mass at the origin!
⇒ Generally true for <u>any</u> spherical mass distribution, if the field is measured completely outside the mass distribution.

③ If $a < r < b$, mass inside $r = \frac{r^3 - a^3}{b^3 - a^3} \cdot M$

$$\vec{g} = -\frac{GM}{r^2} \cdot \left(\frac{r^3 - a^3}{b^3 - a^3} \right) \cdot \hat{r}$$

Note that $\vec{g}(r = a) = 0$ --- agrees with ①
$\vec{g}(r = b) = -\frac{GM}{b^2}\hat{r}$ ... agrees with ②

∴ $\vec{g}$ is continuous everywhere.

Why is $\vec{g}$ continuous?

because mass density is only moderately discontinuous (like a step function)

(→ $\vec{\nabla}\cdot\vec{g} = -4\pi G\rho$ (ρ : mass density)

→ $\vec{g}$ is an integral of ρ

If ρ is only moderately discontinuous
(piece-wise conti like a step ftn)

then $\vec{g}$ will be continuous.)

Next, get Φ from $\vec{g} = -\vec{\nabla}\Phi$

① If $r<a$, $\Phi = \text{const.} = C$

② If $r>b$, $\Phi = -\dfrac{GM}{r} + \text{const} \equiv -\dfrac{GM}{r}$

Just fixing the over-all offset of Φ ← -- by requiring $\Phi(r\to\infty)=0$ (choose to be 0)
(One way to do it)

③ If $a<r<b$, $\Phi = \dfrac{GM}{b^3-a^3}\left(\dfrac{r^2}{2}+\dfrac{a^3}{r}\right) + A$ ← const.

{Note} $\vec{g}$ continuous → Φ must be continuous.
(actually, differentiable!)

Determine A, and C by continuity.

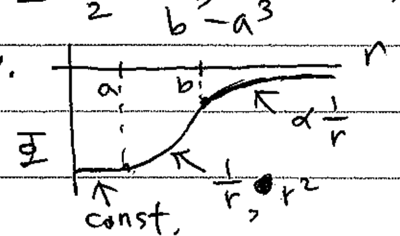
② ... $\Phi(r\to b) = -\dfrac{GM}{b}$ ③ ... $\Phi(r\to b) = \dfrac{GM}{b^3-a^3}\left(\dfrac{b^2}{2}+\dfrac{a^3}{b}\right)+A$

Solving for A we get $A = -\dfrac{3}{2}\cdot\dfrac{GM}{b^3-a^3}\cdot b^2$

$$\therefore ③ \;\cdots\; \Phi = \frac{GM}{b^3-a^3}\left(\frac{r^2}{2}+\frac{a^3}{r}-\frac{3}{2}b^2\right)$$

as $r\to a$, this one goes to $\Phi(r\to a) = \dfrac{3}{2}\cdot\dfrac{GM}{b^3-a^3}\cdot(a^2-b^2)$

This is the value for ① $r<a$ by continuity.



Notes for Lecture 9

Principle of least action

9.1 Principle of least action

Maybe a catchy name: the principle of least action (PoLA). It is also called **Hamilton's principle**.

In this course, this principle is completely equivalent to Newton's laws. From a more general perspective, this principle seems "deeper" or "more elegant" than Newton's law, because it can be applied to other branches of physics as well, like optics, electro-magnetics, and the relativity, when Newton's laws as we learned them becomes impossible, or cumbersome, to generalize. For instance, the well-known **Fermat's principle**, that the light travels in a path that minimizes the time, can be derived from the principle of least action in optics (Homework problem). Also, it is much clearer to see "how quantum mechanics arises" in this view (especially in Feynman's view of quantum mechanics), while, in general, the same task is viewed as impossible if one starts from Newton's laws point of view.

The PoLA means that the following integral, the so-called **action**,

$$S[q(t)] \;=\; \int_1^2 dt\; L(q(t), \dot{q}(t), t)$$

is stationary when $q(t)$ is the actual motion. Let me analyze this sentence one by one. But, first of all, note that the integration range is written as $\int_1^2$, to mean going from t_1 to t_2, not from value 1 to value 2.

1. $S[q(t)]$ defines a **functional**. The square bracket [ ] is used to emphasize the functional nature of S. It means that S takes a function $q(t)$ as input and

returns a number as output. In contrast, a function takes a number and returns a number. **For Hamilton's principle, only those $q(t)$'s for which $q(t_1) \equiv q(1)$ and $q(t_2) \equiv q(2)$ are fixed are considered.** $q(1)$ and $q(2)$ are initial and final positions, in terms of the "generalized coordinate" q. In the integral, $q(t)$ can be any trajectory, physical or hypothetical. Having clearly understood that S is a functional, please do not be surprised, should you see $S(q)$ instead of $S[q]$, when the context is clear that S is a functional.

2. $q(t)$ is a **generalized coordinate**. Here, we only consider one degree of freedom, and so there is only one q. Soon, we will consider many degrees of freedom, in which case we will consider q_i's. The most common example of q is x (or y or z). Or, the angle θ, as in the simple pendulum problem. In general, q can be any function of the linear or the angular coordinates, time and any other parameters of the problem, as long as $L = L(q, \dot{q}, t)$, i.e. the system is fully specified by the generalized coordinate, its time derivative and the time. **So, there will be as many generalized coordinates as the degrees of freedom.** As such, q does not need to have the dimension of length. It can be dimensionless (e.g. angle), or it may have other dimensions (the dimension of the momentum, the angular momentum, the energy, etc). A clever choice of q can make a certain property of the problem obvious (cf. a Homework problem).

3. L is the so-called **Lagrangian**. It is defined as $T - U$, where T is the kinetic energy and U is the potential energy.

4. That $S[q(t)]$ is stationary means the following. Suppose that the true motion occurs along $q_T(t)$. Now, imagine adding a small **variation** $\delta q(t)$ to $q_T(t)$. Consider the subsequent change of S, $\delta S[q_T(t)] \stackrel{def}{=} S[q_T(t) + \delta q_T(t)] - S[q_T(t)]$. That $S[q(t)]$ is stationary at $q_T(t)$ means that

$$\delta S[q_T(t)] = 0$$

As we are just beginning this topic, here we distinguished between the general $q(t)$ and the true motion $q_T(t)$. From now on, however, we will not be making such distinctions, assuming that the context makes it clear when $q(t)$ is an arbitrary[1] one, and when $q(t)$ is the true motion.

Exercise (1) Consider the motion of the free particle, $x = x_0 + v_0 t$. Consider $q(1) = x(1) = 0$ and $q(2) = x(2) = v_0$ (and so we have chosen $t_1 = 0$ and $t_2 = 1$). Show that $\int_1^2 L dt$ is indeed minimum for the true path, by examining the integral for $x = x_0 + v_0 t + f(t)$, where $f(t)$ is any function that satisfies $f(1) = f(2) = 0$. (2) Consider a simply harmonic oscillator with $\omega_0 = 1$. Choose $t_1 = 0$, $t_2 = 2\pi$,

[1] Arbitrary, as long as it is reasonably "nice," e.g. it has a continuous first derivative

$q(1) = x(1) = 0$, $q(2) = x(2) = 0$. In this case, the physical solution can be taken as $\sin t$. Choose *one* differentiable function $f(t)$ (which is not proportional to $\sin t$) that satisfies $f(1) = f(2) = 0$ and show that $S[\sin t + f(t)]$ is ~~greater than~~ $S[\sin t]$, the action integral for a physical motion. [~~Proving this for any function $f(t)$ is possible, also, but it is a much more involved process.~~]

stationary with respect to

9.2 Characteristics of PoLA

PoLA = Hamilton principle. The resulting EOM for L is called the Lagrangian mechanics, as opposed to the Newtonian mechanics.

1. It is a misnomer. Action is not always minimized (for a general potential; the above examples in Exercise are for simple non-general potentials). Rather, $\delta S = 0$ is all we mean and all we need. Minimum, maximum, or saddle point of the action functional – all of these are OK, and do occur. Maxima do not occur in classical mechanics, but it can in other fields (relativistic mechanics).

2. It deals with scalar quantities (action, Lagrangian) rather than vector quantities. It makes life easier.

3. It is equivalent to, but more general than, the Newtonian formulation of classical mechanics. It applies almost everywhere in physics. When it does not, it is generalized easily (the Feynman path integral formalism of quantum mechanics).

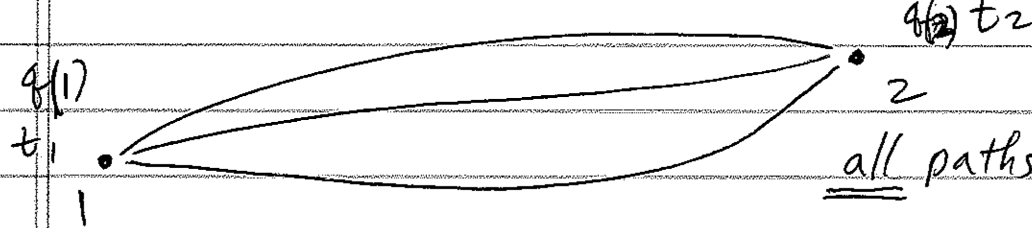
4. It is mathematical. If a free body diagram (Newtonian mechanics) provides forces in full view, the Lagrangian mechanics appears to be just a bunch of math equations, which may seem to "hide all physics under the rug." Newtonian mechanics forces you to think. Lagrangian mechanics may seem to force you to just do the math. Whether this is good or bad is up to you. One should always reflect upon the physics of the solution, though!

5. When bodies are in contact and the motions are constrained, the "force of constraint" (normal force, friction, . . .) comes out of the mathematical formalism, "even if you don't try very hard."

6. The basic formalism applies only when there exists a potential function U (and so all forces are conservative), but the formalism can be extended when there are non-conservative forces, using the concept of Raleigh dissipation function (cf. Landau or any other higher level mechanics text). We will not deal with such a case in this course, except when those non-conservative forces act as "forces of constraint."

2 min. "Vista break"

§. QM (and optics) in a nut-shell



all paths are possible!!

Consider all possible paths and calcultate

$S = \int_1^2 dt\, L$

$e^{iS/\hbar}$ for each random path.

$\hbar$ (planck const. $h/2\pi$)

Sum up all of them (Feynman path integral)

$\Rightarrow$ Get probability amplitude of a particle going from $\underset{(t_1, q_1)}{1} \rightarrow \underset{(t_2, q_2)}{2}$

For a classical particle the path for $\delta S = 0$ gives a singular contribution so that is the only motion possible.

NOT in quantum or optics.

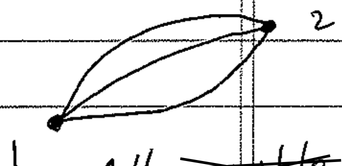
— The End of break —

§. Lagrangian EoM

$$S = \left[\int_1^2 dt\,(T-U) = \right] \int_1^2 dt\, L$$

$$\delta S = \int_1^2 dt \left[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot q}\delta \dot q + \frac{\partial L}{\partial t}\delta t \right] = 0$$

($\frac{\partial L}{\partial t}\delta t$ = 0 by definition of a virtual displacement δq.)



All ~~possible~~ paths

$\delta q(1) = \delta q(2) = 0$

End points are fixed.

$$= \int_1^2 dt \left[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot q}\frac{d}{dt}(\delta q) \right]$$

$$= \int_1^2 dt\, \frac{\partial L}{\partial q}\delta q + \underbrace{\frac{\partial L}{\partial \dot q}\delta q \Big|_1^2}_{=0} - \int_1^2 dt\, \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right)\delta q$$

(= 0 by definition: $\delta q(1) = \delta q(2) = 0$)

$$= \int_1^2 dt \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) \right]\delta q = 0$$

δq is arbitrary $\Rightarrow$ (t_1 —— t_2: non-zero only near a certain value of t ~~a certain value~~ any)

Lagrange (Euler) Equation! $\leftarrow$ $\therefore$

$$\boxed{\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) = 0}$$

at any time!

Generalized to many degrees of freedom $\rightarrow$

$$\boxed{\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) = 0} \quad i = 1, \dots, n \text{ (d.o.f.)}$$

- What if there are forces not expressible as a Lagrangian ??

"generalized force" $\downarrow$

$$\boxed{-\frac{\partial L}{\partial q_j} + \frac{d}{dt}\frac{\partial L}{\partial \dot q_j} = Q_j}$$

e.g. friction, normal force, torque due to friction, etc.

§. Lagragian EoM is equivalent to Newton's eq.

Consider 1D with $U(x)$

$$L = \frac{1}{2} m \dot{x}^2 - U(x)$$

$$x = q$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x}$$

↳ The partial means partial w.r.t. independent variables $x, \dot{x}, t$

(x, $\dot{x}$: implicitly dependent on t; t: any explicit time dependence in L)

$$\frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x} = F(x)$$

$$\therefore \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \quad \text{means}$$

$$m\ddot{x} = F(x)$$

For a more general discussion about the equivalence, read 7.6 of the book.