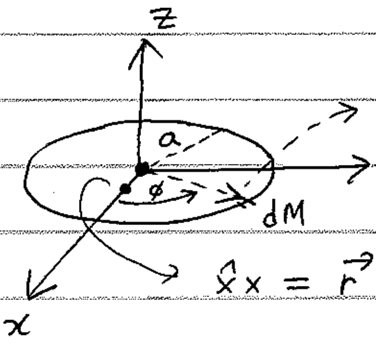
Gravity example (Ex 5,3)

- Consider $\vec{r}$ in the xy plane only.



$$\vec{r}_M = a\cos\phi\,\hat{x} + a\sin\phi\,\hat{y}$$

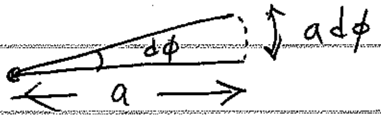
$$\Phi = -\int dM \frac{G}{|\vec{r}-\vec{r}_M|}$$

$$\frac{dM}{a\,d\phi} = \frac{M}{2\pi a} = \text{mass density (linear)}$$

total mass

circumference

$$dM = M\cdot\frac{d\phi}{2\pi}$$



$$|\vec{r}-\vec{r}_M| = \sqrt{(a\cos\phi - x)^2 + a^2\sin^2\phi}$$

$$= \sqrt{a^2 - 2aC_\phi x + x^2}$$

$$\Phi = -\frac{MG}{2\pi}\cdot\int d\phi\cdot\frac{1}{\sqrt{a^2-2aC_\phi x+x^2}}$$

If $|x/a| \ll 1$,

$$\frac{1}{\sqrt{a^2-2aC_\phi x+x^2}} \simeq \frac{1}{a}\cdot\left(1-2C_\phi\frac{x}{a}+\left(\frac{x}{a}\right)^2\right)^{-\frac{1}{2}} \qquad \frac{1}{2}\frac{1}{2}\cdot\frac{3}{2}\cdot 4$$

$$\simeq \frac{1}{a}\cdot\left[1+\frac{1}{4}C_\phi\frac{x}{a}-\frac{1}{2}\left(\frac{x}{a}\right)^2+\frac{3}{2}C_\phi^2\left(\frac{x}{a}\right)^2\right]$$

Integrating,

$$\Phi \simeq -\frac{MG}{a}\left(1+\frac{1}{4}\left(\frac{x}{a}\right)^2\right)$$

One can change $x \to \rho = \sqrt{x^2+y^2}$ considering the ~~sy~~ symmetry of the prob.

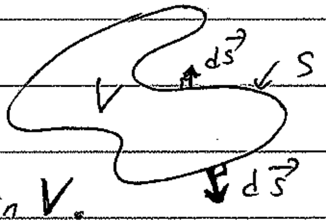
$$\Phi \simeq -\frac{MG}{a}\left(1+\frac{1}{4}\left(\frac{\rho}{a}\right)^2\right) \Leftarrow \begin{pmatrix} z=0 \\ \text{case} \\ \text{only} \end{pmatrix}$$

The origin is an unstable equilibrium point. $\left(\frac{\rho}{a}\right) \ll 1$

Gauss law of gravity

$$\int_V \vec{\nabla}\cdot\vec{g}\, dv = \int_S \vec{g}\cdot d\vec{S} = -4\pi G M$$

- any volume V.
- $S =$ surface of V.
- M is the total mass enclosed in V.



- Any mass outside V does not contribute to the integral.

Proof ? Please read Lecture 7.

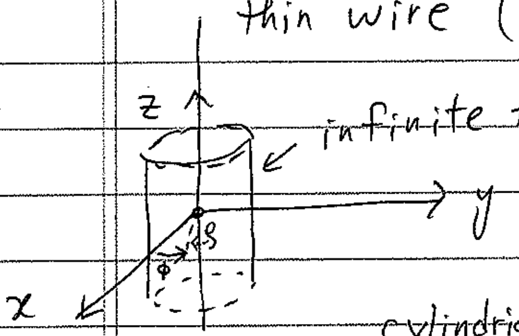
What it does not say: It does not say that masses outside the volume do not contribute to the field in V or on S. It is only the total volume integral or the total surface integral.

Gauss law is highly useful for problems possessing the spherical symmetry or the cylindrical symmetry, ...

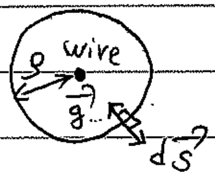
Gravity example #2 Gauss law

thin wire (infinite length)

infinite tube that surrounds the wire



Top view



(ρ, ϕ, z) ← cylindrical coordinates

(ρ = radius of tube)

Considering a tube (infinite length) of radius

$\rho = \sqrt{x^2+y^2}$

and using Gauss law for it

$$\int_{\text{tube}} \vec{\nabla}\cdot\vec{g}\, dv = \int_{\text{tube surface}} \vec{g}\cdot d\vec{s} = - g \int \int_0^{2\pi} dz\, d\phi\, \rho$$

length of tube or wire $L \to \infty$ ‖

↑ the magnitude of $\vec{g}$ constant on the tube due to the cylindrical symmetry

$$= - g L 2\pi \rho$$

$$= - 4\pi G M$$

M: total mass of the wire ($\to \infty$)

L: length of the wire ($\to \infty$)

$A \equiv M/L$ = linear mass density of the wire (finite) = mass per unit length

$$g = 2GA/\rho$$

$$\vec{g} = - 2GA\,\hat{\rho}/\rho$$

− sign means $\vec{g}$ towards the origin

$$\Phi = + 2GA \ln \rho$$