Notes for Lecture 7

Chaos and Gravity

In the last lecture, we discussed main aspects of nonlinear systems. Subharmonic and superharmonic resonances, amplitude jump and hysteresis are some of the novel features of nonlinear systems.

In considering the periodicity and the loss of periodicity and a possible emergence of a chaotic behavior, the Poincare-Dixon theorem sheds some light. It says that if the dimension of the phase space is either less than 3 and if there is no external driving force, then the motion can never be chaotic. For instance, a simple pendulum can never be chaotic, without an external force. Also, a two body problem like the earth pulled by the sun is never chaotic, as, the effective phase space dimension in this case is just one. On the other hand, the phase space dimension of a three body problem or a double pendulum is greater than 2, and chaotic motions do occur for such a problem.

7.1 Chaos

Chaotic systems can be defined as showing an extreme sensitivity to the initial condition. One commonly used criterion for chaos is the positive **Lyapunov exponent**. It is defined as λ in

$$\delta Z \approx e^{\lambda t}\varepsilon$$

as $t \to \infty$ and $\varepsilon \to 0$, where ε is the initial separation of the two initial conditions, and δZ is the final separation ($t \to \infty$) of the two initial conditions. If $\lambda > 0$, then the exponential factor diverges and so δZ will be finite, if we wait long enough, for any infinitesimal ε.

Note that in the above equation, in order to get $\delta Z \sim 1$, the waiting time $t \sim -\frac{1}{\lambda}\ln\varepsilon$ is necessary. So, even when ε is a small number, the waiting time will be modest, if $1/\lambda$ is of the order of the inherent period of the system (typically the case). By the same token, even when ε is reduced by a huge factor such as 10^4, the increase of the waiting time for observing the chaotic behavior is only mild, due to the logarithmic ratio 4 (detailed calculations left for students' work! – some calculator work seems necessary to go beyond a crude estimate).

The above definition does not specify whether we consider a bound motion or an unbound motion. For the study of chaos, we will consider a bound state motion only, that is the motion is confined to a finite region in the phase space. For linear systems, $\lambda \leq 0$ for a bound motion (see Lecture Note 5). For nonlinear systems, $\lambda > 0$ *can* occur.

The above definition of chaos means that no matter how close two initial conditions are in the phase space, sooner or later the two motions will be completely separated in the phase space with a finite distance between them. [This is quite different from the behavior of a linear system, where the two nearly identical initial conditions will result in two nearly identical motions at *any time*. This was discussed in Lecture Note 5.] And, they are non-periodic motions. And, they are bound motions... If it seems a bit hard to imagine what kind of motion that is, then it is probably because it is hard to imagine. What happens in chaotic motions is repeated "stretching" and "folding" that remaps points of a bound region to the same region. This results in very complicated self-similar dense structures. Such a structure is called a fractal.

Science of ... chaos?

This may sound a bit of an oxymoron. How can there be a science of chaos, if, by definition, a chaos means a hopelessly unpredictable situation.
Oddly enough, in modern physics, terms like "chaos," "frustration," "disorder," "complexity" and "symmetry breaking" have come to assume plenty of importance, although when you first hear them they might make you frown. Indeed, we are learning, in various contexts, that something novel, something genuinely creative, tends to happen in the vicinity of "chaos" or a genuinely unpredictable situation. They may include interesting pattern formations due to self-organization (how do clouds form?), or fundamental physics associated with symmetry breaking (the superconductivity and the Higgs boson physics).

Poincaré section

A chaotic motion traces a path in the phase space in a very complicated way. Consider a motion of a single particle in one dimension. Then the phase space is two dimensional, and including time, we have a three dimensional space to consider. As the phase path progresses in the phase space, the result can be a very complicated one. Poincaré noted that if a discrete set of times are sampled "wisely," then the resulting map can make it possible to distinguish between periodic motions that could admit analytic solutions, or chaotic motions that do not. There is no general method known to construct a Poincaré section, though, and a problem specific method is applied, although there exist methods developed for certain classes of problems. For a driven non-linear 1D oscillator, the Poincaré section can be easily defined as a "stroboscope" picture in phase space, taken with the period of the driving force. In a chaotic regime, a fractal pattern characterizes the Poincaré section (Figure 4-19 of the book).

Attractor

A set of point(s) to which the motion converges for a dissipative system. This can be a fixed point or a limit cycle (non-chaotic motion), or a strange/chaotic attractor (chaotic motion). Strange attractors are fractals. An example of a strange attractor is the Lorenz attractor, governed by the following equation.

$$\begin{aligned} \frac{dx}{dt} &= \sigma(y - x) \\ \frac{dy}{dt} &= x(\rho - z) - y \\ \frac{dz}{dt} &= xy - \beta z \end{aligned}$$

7.1.1 Maps

A dynamical system can be defined as a discrete map mathematically. In this case, the dynamics is given as a series: $x_0 \to x_1 \to x_2 \to \cdots \to x_n \to \cdots$.

The Lyapunov exponent, λ, for the discrete map is defined as

$$d_n = \exp(\lambda n)\varepsilon$$

in analogy with the continuous case in page 1. Here $d_n = |x_{1,n} - x_{2,n}|$ and $\varepsilon = |x_{1,0} - x_{2,0}|$, where the subscripts 1 and 2 mean two instances of the map with minutely different initial conditions, $\varepsilon \to 0$. It is left for readers' work to show that (e.g. see page 176 of the textbook):

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \ln \left| \frac{df(x_n)}{dx_n} \right|$$

An example of a map is a logistic map.

$$x_{n+1} \;=\; \alpha x_n(1 - x_n)$$

Here, $0 \le x_n \le 1$, $n = 0, 1, 2, \cdots$.

First of all, for discrete systems, the Poincare-Bendixon theorem does not apply, and this type of one-dimensional map *can* shows chaotic behavior – and it certainly does!

The logistic map can be a crude model of a biological system in an isolated environment with finite resources. If there is a small population to begin with, then the system will start reproducing and grow (assuming $\alpha > 1$), as the above equation behaves as $x_{n+1} \;\approx\; \alpha x_n$ if $x_n \ll 1$. However, if the population is too large, then it will consume all finite resources and the population will collapse. This is the behavior at the other end $x_n \to 1$, where $x_{n+1} \;\approx\; \alpha(1-x_n)$. n may be thought of as some unit of time, like a year. This model has an interesting behavior. First of all, an obvious solution does emerge. A stable and steady population that represents the equilibrium between the finite resources and the population's tendency to reproduce. However, as the parameter α is tweaked up, more interesting behaviors appear. A sort of "bistable" situation occurs, where the population alternates between two values year by year! This is called the "bifurcation." It is analogous to the periodicity doubling in non-linear oscillators that we discussed. Each of those bifurcated points can bifurcate again, which means the appearance of period 4. This process can continue, and pretty soon, a chaotic behavior emerges.

Figures 4.22, 4.23, and 4.25 are to be understood, in the context with these descriptions.

Notice that in Figure 4.25, the Lyapunov exponent becomes 0 whenever the bifurcation occurs. It is as though whenever the bifurcation occurs, the system is "knocking on the door of the chaos." As the bifurcation process continues, the system finally becomes chaotic (the Lyapunov exponent becoming greater than 0) as the α value becomes greater than $\approx$ 3.57. However, when α value increases more, the system shows a return to non-chaotic behavior in some finite value ranges of α.

An interesting characteristic of a chaotic map such as the logistic map is the so-called "Feigenbaum's number," which is defined as $\lim_{n\to\infty} \delta_n \;=\; \Delta\alpha_n/\Delta\alpha_{n+1}$, where

$\Delta\alpha_n = \alpha_n - \alpha_{n-1}$, and α_n is the value of α for the n-th bifurcation point [Example 4.2 of the book].

7.2 Gravity

7.2.1 Newton's law of gravity

For two bodies, there is an attractive force of the magnitude

$$F = G\frac{Mm}{r^2}$$

and the direction which is parallel to the line joining the two bodies. Here, M, m are the masses of the two bodies. r is the distance between them.

7.2.2 Gravitational field

Field is an important modern concept. It does away the "action at a distance," which Newton himself had a hard time believing (and so did Einstein).

Consider a body of mass at M found at some point. Let us conveniently take that point to be the origin. Then, we say that the gravitational field, $\vec{g}$, at position vector $\vec{r}$ due to this mass M is

$$\vec{g} = -\frac{GM}{r^2}\hat{r}$$

where $r = |\vec{r}|$, and $\hat{r}$ is the radial unit vector $\hat{r} = \vec{r}/r$. Observe that taking the position of mass M was purely for convenience. In general, we can change $\vec{r} \to \vec{r} - \vec{r}_M$, and $\hat{r} \to (\vec{r} - \vec{r}_M)/|\vec{r} - \vec{r}_M|$, where $\vec{r}_M$ is the position of mass M, and everything is good.

Then, the force that another mass m feels due to M is given by

$$\vec{F}_m = m\vec{g}$$

Two comments. (1) We are defining $\vec{g}$ generally here, not just the Earth gravity near its surface. In general, $\vec{g}$ is position dependent, not constant. (2) The definition of a field is a mathematical triviality, at this level. But, imagine that the field is some sort of a real thing that connects two massive bodies! The concept of the field is a big deal, while the particles we think are responsible for the gravitational field (gravitons) haven't been detected by any human scientific equipment yet (compare this situation with photons which are responsible for the electromagnetic field).

7.2.3 Gravitational potential

Let us look at the mathematics of the field a bit: $\vec{g}(\vec{r}) = -GM\hat{r}/r^2$. This is a conservative field (the proof of this is very similar to that of Homework 1, 9(c)). Which means two things:

$$\vec{\nabla} \times \vec{g} = 0$$
$$\vec{g} = -\vec{\nabla}\Phi(\vec{r})$$

Here, Φ is related to the potential energy U via $U = m\Phi$, where m is the other mass that interacts with mass M. As in Homework 1, 9(c), the gravitational potential Φ can be obtained as

$$\Phi(\vec{r}) = -\frac{GM}{r}$$

Or, generally, if the mass M is not at the origin, but at $\vec{r}_M$:

$$\Phi(\vec{r}) = -\frac{GM}{|\vec{r} - \vec{r}_M|}$$

Let us consider a simple fact. If there are multiple bodies, then the total force is obviously the addition of all forces. Each force can be considered as coming from a potential field. It then follows that the potential field is also additive. This is due to the linear operator nature of the $\vec{\nabla}$ operator that connects the potential and the field. So, for a gravitational field that arises from multiple bodies,

$$\Phi(\vec{r}) = -G\sum_i \frac{M_i}{|\vec{r} - \vec{r}_{M_i}|}$$
$$\vec{g} = -G\sum_i \frac{M_i}{|\vec{r} - \vec{r}_{M_i}|^3}(\vec{r} - \vec{r}_{M_i})$$

For a continuous distribution of masses, these change to the integral:

$$\Phi(\vec{r}) = -G\int \frac{dM}{|\vec{r} - \vec{r}_M|}$$
$$\vec{g} = -G\int dM\frac{\vec{r} - \vec{r}_M}{|\vec{r} - \vec{r}_M|^3}$$

7.2.4 Gauss law, Poisson equation

Let us consider a volume integral

$$\int_V dV\vec{\nabla}\cdot\vec{g} = \int_S d\vec{S}\cdot\vec{g}$$

The above equality of the volume integral of a divergence of a vector field and the surface integral of the vector field is called **Gauss's theorem**. Very important. Here, S is the surface area of the volume V, and $d\vec{S}$ is a small area element vector. The magnitude of $d\vec{S}$ is the area of the area element, and the direction of it is normal to the area element and towards the outside of the volume.

Consider the simplest case first. Assume $\vec{g} = -GM\hat{r}/r^2$, and the volume V is a sphere of a radius R, centered at the origin. The integral is then

$$\begin{aligned}\int_V dV \vec{\nabla}\cdot\vec{g} &= \int_S d\vec{S}\cdot\vec{g} \\ &= -\frac{GM}{R^2}\int_0^{\pi} d\theta \int_0^{2\pi} d\phi \sin\theta\ R^2 \\ &= -GM\int d\Omega \\ &= -4\pi GM\end{aligned}$$

Here, $d\Omega$ is the infinitesimal solid angle[1]

$$d\Omega \stackrel{def}{=} \frac{d\vec{S}\cdot\hat{r}}{r^2}$$

subtended by the area element $d\vec{S}$ at the origin. Using the concept of the solid angle, the above integral immediately generalizes to any volume, which may or may not enclose the origin:

$$\begin{aligned}\int_V dV \vec{\nabla}\cdot\vec{g} &= \int_S d\vec{S}\cdot\vec{g} \\ &= -GM\int d\vec{S}\cdot\hat{r}/r^2 \\ &= -GM\int d\Omega\end{aligned}$$

Note that whenever the volume V encloses the origin, then it is $\int d\Omega = 4\pi$, while if V does not enclose the origin, then $\int d\Omega = 0$.

$$\begin{aligned}\int_V dV \vec{\nabla}\cdot\vec{g} &= -4\pi GM \qquad \text{if } V \text{ encloses the mass } M\text{, the source of } \vec{g}, \\ &= 0 \qquad\qquad\qquad \text{if it does not,}\end{aligned}$$

[1]This is the general definition of the infinitesimal solid angle for an area element $d\vec{S}$ which is at the position vector $\vec{r}$. Note that it can be positive or negative depending upon whether the normal vector of the area element is pointing away from the origin or towards it. This should not be surprising, given the fact that the "linear" angle also has a sign. In the spherical coordinate, the volume element $dV = r^2 \sin\theta\ drd\theta d\phi = r^2 drd(\cos\theta)d\phi$. This is **always** equal to $dV = r^2 drd\Omega$. This may be used as an alternative definition of $d\Omega$, if you like, and is equivalent to the above definition.

for *any volume* V.
What if the mass M is not placed at the origin? In that case, $\int dV \vec{\nabla} \cdot \ldots = \int dV_M \vec{\nabla}_M \cdot \ldots$ by a mere translation of the coordinate vectors, where the subscript M means the coordinate system whose origin is at $\vec{r}_M$. Therefore, the above result is valid even if M is displaced from the origin. [**Note 1**: This simply means that if we shift the position of the mass and the volume of integration at the same time, the integral should not change at all. This is obvious since we are free to choose the origin at any point that we like. **Note 2**: The key point to remember here is that no matter how we deform the volume it will remain invariant as long as the mass stays inside the volume, if the initial volume enclosed the mass, or the mass stays outside the volume, if the initial volume did not enclose the mass. For instance, suppose we start from a sphere with a mass M at the origin of the sphere. If we shift the sphere a little so that now the mass M is off-center with respect to the sphere, the above integral remains invariant as long as M continues to stay inside the sphere.]

Therefore, the above result then immediately generalizes to the case when there is any distribution of masses, not just one mass.

So, what we have is the generalization of the above result:

$$\int_V dV \vec{\nabla} \cdot \vec{g} \; = \; \int_S d\vec{S} \cdot \vec{g} \; = \; -4\pi G M$$

V is *any volume*.
M is the total mass enclosed by V, including 0.
$\vec{g}$ is the total gravitational field, due to *all masses* around, not just M.

Since V is any volume, it can be taken to be dV. Then, $M = \rho(\vec{r})dV$, where $\rho(\vec{r})$ is the mass density. Then, we have

$$\vec{\nabla} \cdot \vec{g} \; = \; -4\pi G \rho(\vec{r})$$

These two boxes above contain an extremely useful law of physics: the **Gauss's law** for the gravity. Gauss's law is applicable to Newton's law of gravity and Coulomb's law of the electrostatic force.

Gauss's law can be re-written in terms of Φ,

$$\vec{\nabla}^2 \Phi = 4\pi G \rho(\vec{r})$$

This is the **Poisson's equation** for the potential Φ. It is a kind of differential equation, like the SHO ODE that we solved.

For a point mass at $\vec{r}_M$, $\rho(\vec{r}) = M\delta(\vec{r} - \vec{r}_M)$, where $\delta(\vec{r} - \vec{r}_M)$ is the Dirac-delta function in 3 dimensions, note that we know the solution Φ already. It is $-GM/|\vec{r} - \vec{r}_M|$. So, we can write

$$-\vec{\nabla}^2 \frac{1}{4\pi|\vec{r} - \vec{r}_M|} = \delta(\vec{r} - \vec{r}_M)$$

That is, $-\frac{1}{4\pi|\vec{r} - \vec{r}_M|}$ is the **Green's function** of the Poisson equation (cf. Homework 3, Problem 2).

7.2.5 Meaning of the gradient

Suppose you make a plot of equipotential lines, i.e. a collection of curves, which satisfy Φ = constant. Where does the force field, $\vec{g}$ point? Due to the nature of the gradient, $\vec{g}$ always points perpendicular to the equipotential line. This does not uniquely determine the direction of $\vec{g}$. It could point along the direction in which the potential increases, or the direction in which the potential decreases. As $\vec{g}$ is the negative gradient of Φ, $\vec{g}$ points towards the direction in which Φ decreases. Look at Figure 5.8 of the textbook, and figure out which way the force field is pointing at some points. Also, note that the force field is greater in magnitude where the equipotential lines are dense.