

Notes for Lecture 6

Non-linear systems

In the previous lecture, we covered the resonance phenomenon. This was the subject of a driven oscillator. We also covered some grounds of non-linear oscillator physics. In this lecture, we will discuss non-linear systems a bit more, with a focus on chaos, a subset of non-linear phenomena.

Before that, a few words about the resonance phenomenon.

For a damped SHO, when $\beta > 0$, the complementary function x_c always decays (and so it is only part of transient response). In contrast, the particular solution for a sinusoidal driving force is a *steady state* solution that oscillates in time without decaying. Question: what if there is *no* damping? First, the complementary solution is now oscillatory, of course, as it describes a free SHO. Second, the particular solution can be obtained by taking the $\beta \rightarrow 0$ limit on our solutions already obtained in the previous lecture, except on resonance. On resonance ($\omega = \omega_0$), our particular solution of the last lecture, $D \cos(\omega_0 t - \delta)$, has the same functional form (that is, it can be written as $A \cos(\omega_0 t) + B \sin(\omega_0 t)$) as the complementary solution, and so it is no longer valid as a particular solution. Instead, the particular solution now features a linearly increasing amplitude. Which makes sense, since the energy is continually put into the system by the driving force, while there is no dissipative mechanism. Here is the mathematics of it. We need to solve $\ddot{x} + \omega_0^2 x = A \cos \omega_0 t$. By assuming (for the complex plane version of this EOM), $x(t) = g(t) \exp(i\omega_0 t)$, you should be able to show that $g(t) = \frac{-Ai}{2\omega_0} t$ does a perfect job for the particular solution.

6.1 Amplitude jumps and hysteresis

For a linear system, we derived that

$$D = \frac{F_0/m}{\sqrt{(\omega^2 - \omega_N^2)^2 + 4\beta^2\omega^2}}$$

where ω is the driving frequency and ω_N is the natural frequency (formerly known as ω_0).

In the last lecture, we learned that when non-linear potentials are treated perturbatively, then the natural frequency changes according to

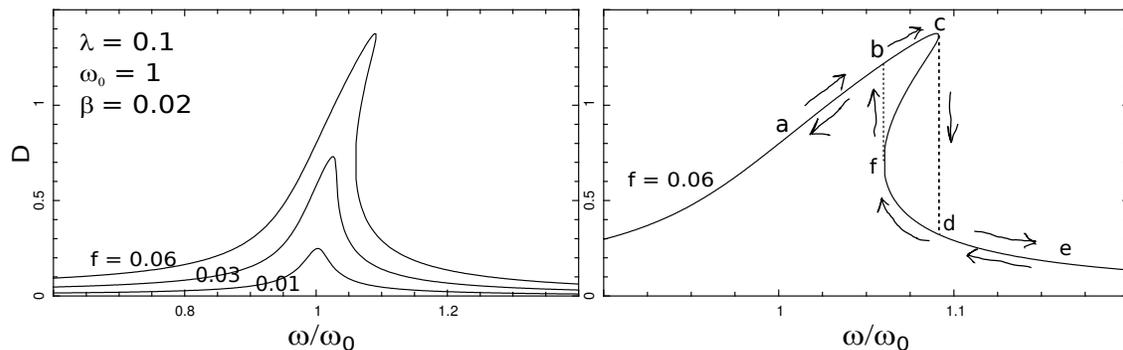
$$\omega_0'^2 \approx \omega_0^2 + \lambda D^2$$

Here, λ is a small number that occurs due to the cubic (softening) potential and the quartic (hardening) potential and so on.

This ω_0' is what we need to plug in for ω_N above, as that is the natural frequency of the system in the presence of the non-linear effects. So, for a non-linear system, we have the following equation

$$D = \frac{F_0/m}{\sqrt{(\omega^2 - \omega_0^2 - \lambda D^2)^2 + 4\beta^2\omega^2}}$$

This is now a cubic equation for D^2 , and so it generally can have three roots for D ! Here are some diagrams that are calculated as a function of $f = F_0/m$.



As expected the resonance frequency increases as f , and thus D , increases. However, a qualitatively new feature appears. The resonance profile is no longer single valued near the peak, if the driving force is sufficiently large! What this means is the following. As the value of ω is ramped up, the system will respond along the

a-b-c line, at which point suddenly the amplitude must jump down to d, and then follow the path d-e. If the frequency is ramped down from high frequency, then the amplitude of the oscillator follows the path e-d-f-b-a, where f-b is a sudden jump of the amplitude. This is the phenomenon of amplitude jumps and hysteresis. As the phase shift is given by

$$\delta = \tan^{-1} \left(\frac{2\beta}{\omega_N^2 - \omega^2} \right)$$

it will also show jumps and hysteresis as D , since $\omega_N^2 \approx \omega_0^2 + \lambda D^2$ (see Figure 4.12 of the textbook).

6.2 Super-harmonic and sub-harmonic resonance

The above discussion is not all the story, of course. In the discussion above, we have tacitly assumed that the system acts as a linear oscillator except for the amplitude dependent natural frequency. But, we know that a higher harmonic generation is the characteristics of the non-linear operator. So, in a nonlinear system, the resonance response does not only appear at the natural frequency but also at higher harmonics of it. This has two implications. First, for a given fundamental resonance at $\approx \omega_N$, higher harmonic responses will be co-generated. This is the basis of the higher harmonic generation in non-linear optics. Second, there will be resonance lines at low frequencies, $\approx \omega_N/n$. This is the so-called **super-harmonic resonance**. What happens is that when a driving force with frequency $\omega = \omega_N/n$ is used, then due to the non-linearity, higher harmonics of ω will be induced. Eventually, the higher harmonic generated will match the natural frequency of the system ω_N , which induces a resonance response.

The **sub-harmonic resonance** is the opposite concept, in which the driving force with frequency ω produces a response at frequencies ω/n , and thus the resonance occurs at integer multiples of the fundamental frequency ω_N . How is this possible? Numerically, this effect starts to show when the system is driven strongly so that the amplitude response is large, and thus the non-linear effect is no longer small. So, it is plausible to qualitatively consider the origin of this effect, as a perturbation at the opposite limit, i.e. the perturbation theory when the non-linear effect dominates. To be very specific let us assume, by considering the cubic non-linear potential, that

$$\alpha x^2 + f(t) = \ddot{x} + 2\beta\dot{x} + \omega_N x$$

and $|\alpha|D^2 \gg D$ (where D is the amplitude scale) so that the RHS can be considered as a perturbation. Such a large amplitude situation is appropriate for systems close to or in a chaotic regime. With $f(t) \propto e^{i\omega t}$, $x(t) \propto e^{i\omega t/2}$ in the zero-th order

approximation. Plugging in this solution to the RHS to get the next order solution is equivalent to adding an external force term with frequency $\omega/2$. This would lead to the first order solution, which would then be governed by two frequencies: ω and $\omega/2$. Then, the second order solution will have three frequencies ω , $\omega/2$ and $\omega/4$. One can see that this process can continue indefinitely. Had the quartic and higher terms of the potential energy been added, then we would also have other frequencies such as $\omega/3$.

It can be seen that the sub-harmonic generation of an original frequency “input” to the system (ω), will lead to a longer and longer period (a cascade of period doubling or tripling), as ω become fractionalized (ω/n). Eventually, the system may become completely non-periodic and chaotic. The super-harmonic generation will not do such a thing, as the overall period remains the same in that case, when ω becomes doubled or tripled or more. Both effects (sub-harmonic and super-harmonic effects) will, of course, grow as the non-linearity of the system becomes more important.

6.3 Plane pendulum – fixed point, separatrix

A plane pendulum is one of the simplest non-linear systems. It is *not* a chaotic system at all, but it does have an interesting feature to note: separatrix. Also, the notion of the fixed point can be learned.

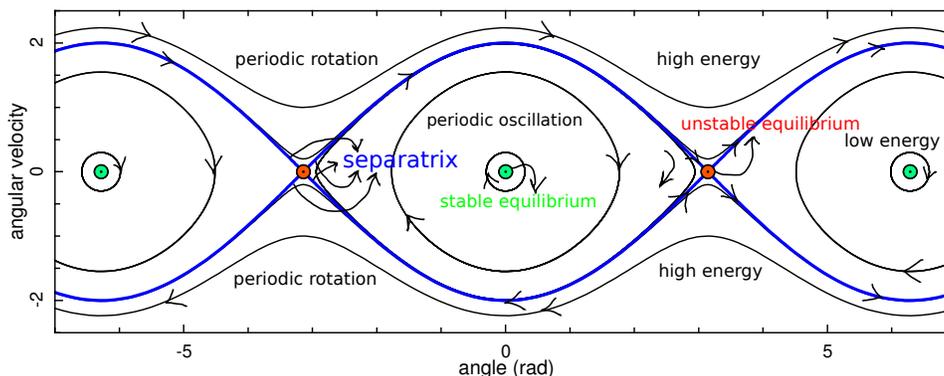
Below are some examples of the motions represented in the phase space. First, we note that $\theta = 0$ modulo 2π is a stable equilibrium point (green points) and $\theta = \pi$ modulo 2π is an unstable equilibrium point (red points). Mathematically, the equilibrium points (stable or otherwise) are called “**fixed points**,” as the system will stay there forever if found there. Physically, of course, this statement (“will stay there forever”) is true only for the stable equilibrium point, as any “noises” or “fluctuations,” thermal or otherwise, found in real systems will drive the system away from the unstable equilibrium point.

Around the stable equilibrium point, the motion is represented by a circle if ω_0 is taken to be 1 (close to the equilibrium point; this is the SHM) or a closed path (not so close to the equilibrium point). These correspond to $E < mgl$, assuming that we take U to be $-mgl \cos \theta$.

If $E > mgl$, then the motion corresponds to a rotation.

When $E = mgl$, then the phase diagram of the motion passes through the unstable equilibrium point. The diagram (blue curves) is an example of the so-called “**separatrix**.” A separatrix is a phase space diagram that separates different types of

motions, and it always passes through unstable equilibrium point(s). In the current example, any point found inside the separatrix corresponds to an oscillation, while any point found outside it corresponds to a rotation, while both kinds of motion are periodic.



On the separatrix, the motion is peculiar. If the system is found on the separatrix but not on the unstable equilibrium points, then the system is either approaching the unstable equilibrium point (like one of the arrows moving towards the red point) or departing it (like one of the two arrows moving away from the red point). The approach takes an infinite amount of time. The departure, on the other hand, will take a finite amount of time. However, the eventual fate of this departing motion is another ever-lasting approach to the unstable point. The departure and the approach are time-reversed states of each other.

6.4 Periodic motion to non-periodic motion

We have sketched above how a system might become non-periodic (sub-harmonic generation). However, this is still an active area of research and no general analytical method is known for predicting whether a given system will act chaotically. Instead, numerical methods play a central role in studies of non-linear systems. However, it is possible to identify an important category of simple motions that will *not* show chaotic behaviors.

We learned in a previous lecture that **all 1D bound motions, governed by a potential function $U(x)$, are periodic**. Clearly, a periodic motion is not chaotic. So, let us think about variations of this statement.

Here, we consider a potential $U(x)$ with a finite number of equilibrium points in

any finite interval of x . Also, what we are interested in is the **long-term behavior** ($t \rightarrow \infty$), not the short-term behavior (transient behavior near the initial time).

- What if a dissipative force is added?
If the force is dissipative, then the system would lose its energy over time. Eventually, it will be driven to a stable equilibrium point. So, a long term behavior is a trivial **fixed point** behavior. There is no motion in the long run. You might say that this an extremely boring, and non-chaotic “motion.”
- What if a sinusoidal external driving force is added as well as a dissipative force, while the system is **linear**?
This is the driven SHO problem that we already solved. In this case, a nice steady state solution with the same periodicity as the driving force emerges. So, a **periodic motion** is the long term behavior.
- What if a sinusoidal external driving force is added as well as a dissipative force, while the system is **non-linear**?
We haven’t dealt with any problem of this kind yet. In general, it is hard to know what will happen, and indeed a **chaotic behavior** can occur. And, this is not surprising given our arguments (sub-harmonic generation) above.
- How about in higher dimensions?
Indeed. In higher dimensions, a mechanical problem, even without the dissipation and the external driving force, is generally very hard to solve. A **chaotic behavior** can occur at large amplitudes when the non-linearity becomes important! An example is a double pendulum, which has two degrees of freedom instead of one. More precisely speaking, what is important is that the problem of the double pendulum is a three dimensional problem in phase space (four to start with, energy conservation reduces the dimension by 1), higher than two (see the box below for the Poincaré-Bendixon theorem). Another example is the famous three body problem, which begot the concept of chaos in the first place (through the famed work of Poincaré).
- What if there is a dissipative force and an “anti-dissipative” force?
An anti-dissipative force is one that adds energy, instead of subtracting it. The van-der Pol equation is one such example.

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0, \quad \mu > 0$$

This model has a self-regulating dissipation and “anti-dissipation” mechanism so that eventually any solution will converge to a periodic bound motion, which is called a **limit cycle**. There is obviously a trivial fixed point at $x = 0$ and $\dot{x} = 0$. These possible behaviors can be understood within the “**Poincare-Bendixon**” theorem. See the box below.



Poincaré-Bendixon theorem

Consider a differential equation $d\vec{r}/dt = f(\vec{r})$ where \vec{r} is the position vector in a 2D plane,^a $\vec{r} = (x, y)$. This theorem^b states that the bound state solution of this equation converges to either an equilibrium point, a limit cycle or a separatrix. This means no chaotic behavior for a one dimensional closed mechanical system (closed in the sense that there is no external force $f(t)$ or any explicit time dependence on the force; dissipative forces are still allowed). Note that this theorem does not rule out a chaotic behavior of discrete dynamical systems (so-called “maps” such as the logistic map, Arnold’s map, and the baker’s map), nor does it rule out a chaotic behavior if the system is driven by an external force (so that f acquires an explicit time dependence; driven simple harmonic oscillator and driven van der Pol equation are examples of chaotic systems).

^aor a cylinder, or a sphere, etc.

^b Introduction to Applied Nonlinear Dynamical Systems and Chaos Texts in Applied Mathematics, 2003, Volume 2, 117-121

Limit cycle

Consider a 2D phase space. If there is a closed trajectory in the phase space to which other trajectory converges to as $t \rightarrow \infty$ or $-\infty$ is called a limit cycle. This behavior is exhibited in some non-linear systems. The van der Pol oscillator is a good example of such a system.