

Notes for Lecture 2

Newton's laws (continued)

Previously, we have introduced particles, dimension, degrees of freedom, vectors, scalars, Newton's laws, inertial reference frames, and how Newton's equation of motion (2nd law) is *always* solvable, in principle,¹ by a computer.

Did you notice that the expression “**integrable**” is used for Newton's equation? That means “analytically integrable” since Newton's equation is always integrable numerically. Indeed, in this lecture, we will examine some integrable Newtonian equations of motion (EOMs).

Did you also notice that there is a circular structure to Newton's laws? (i) Consider a particle with mass m_1 . No acceleration? That means no force. Acceleration? That means force. If it sees another particle with mass m_2 , then it exchanges 3rd law pair forces. (ii) Now, group these two masses together and call it a “particle” with mass M_1 . Consider it as a point. No acceleration of M_1 ? It means no *external* force! This law of inertia is now a *derived* property that depends on the last step of (i). Acceleration? Must be an external force. If M_1 sees another particle with mass M_2 , then they will exchange 3rd law pair forces. This is again a derived property, now. (iii) Now, group these two masses, M_1 and M_2 as a “particle.” Consider it as a point. The law of inertia for this new particle is again guaranteed by the last step of (ii). And on and on.

While this circular structure is a sure sign for the self-consistency of the theory, it also means a certain lack of starting point. Indeed, the initial “particle” in the above reasoning is also a macroscopic object consisting of many fundamental particles of Nature. If this initial particle is divided up too finely, Newton's laws break down.

¹Well, there is a serious question of how big a computer you need and how much time you need, when many particles are involved. That is for discussion at another time.

(See my note on “Nature of Physical Laws.”) It is important to keep in mind that all physics theories are based on experiments.

2.1 Example 2.4 of text

Air resistance $F = -kmv$, $k > 0$. **One dimensional motion.** $x(t), v(t)$?

Equation of motion:

$$m\dot{v} = -kmv$$

We have a rather happy situation here. It is a first order differential equation for v !

This is easy to solve.

$$\begin{array}{ll}
 m\dot{v} = -kmv & \\
 \frac{dv}{dt} = -kv & \text{divide by } m, \text{ use Leibniz notation} \\
 \frac{dv}{v} = -kdt & \text{multiply both sides by } dt \\
 \frac{dv(t')}{v(t')} = -kdt' & \text{change variable } t \rightarrow t' \\
 \ln v - \ln v_0 = -kt & \text{integrate from } t' = 0 \text{ to } t' = t \\
 \ln(v/v_0) = -kt & \ln(A/B) = \ln A - \ln B \\
 v = v_0 \exp(-kt) & \ln X = Y \rightarrow X = \exp Y
 \end{array}$$

This is a simple, but very important result. **If there is an air resistance that is linear in velocity, then the velocity will decay exponentially.**



Exponential decay

The equation of the form $dv/v = -kdt$ occurs ubiquitously in physics (and elsewhere). If v is not the velocity, but the number of a certain kind of particles with a finite lifetime $\sim 1/k$, (the symbol \sim means “on the order of” or “up to a multiplicative constant”) then this equation describes the population decay of that particle. As such, the Rutherford radioactive decay law $N(t) = N(0) \exp(-kt)$ can be understood with a similar microscopic mechanism: a radioactive particle has a certain probability to decay, kdt , in the small time interval, dt . Such a decay happens for naturally unstable particles (e.g. muons), or for virtually all excitations inside materials, e.g. the electron-hole excitation in solar cells, although in the latter case, the decay is followed by emission of visible light, not γ rays!

Note that the dimension of k is $1/T$ as can be seen easily from $F = -kmv$. So it makes a perfect mathematical sense that kt appears together in the exponential function. In the argument of a transcendental function, only a dimension-less quantity can appear, such as kt here.

Namely, in this problem $1/k$ is a **time scale**, which determines the decay behavior. k is called a **decay constant**.

x is easy to get, since the exponential is readily integrable.

$$x - x_0 = \frac{v_0}{-k} (\exp(-kt) - \exp(-k \cdot 0)) \quad \because \int \exp(-kt') dt' = \exp(-kt) / -k \quad (2.1)$$

$$x = x_0 + \frac{v_0}{k} (1 - \exp(-kt)) \quad (2.2)$$

$$= x_\infty - \frac{v_0}{k} \exp(-kt) \quad x_\infty \stackrel{\text{def}}{=} x_0 + v_0/k \quad (2.3)$$

As the result of the air resistance, the range of motion is limited to a finite interval, and the particle converges to x_∞ never reaching it. The finite range of motion, $x_\infty - x_0 = v_0/k$, can be understood easily this way also. Due to the fractional decrease in the velocity $dv/v = -kdt$, the distance travelled during each small time interval dt is slightly less than the distance travelled during the previous interval of the same duration, by the factor $1 - \varepsilon$ where $\varepsilon = dv/v$. The total distance travelled is then given as a simple geometric series: $v_0 dt [1 + (1 - \varepsilon) + (1 - \varepsilon)^2 + \dots]$ where $\varepsilon = dv/v = kdt$. This is the range of motion $= v_0 dt / (1 - (1 - \varepsilon)) = v_0 dt / \varepsilon = v_0 dt / (kdt) = v_0/k$.



Distance and velocity

In an accelerated motion, we often ask the question about the relationship between the distance and the velocity. In this example, that relationship is obtained easily by eliminating t from our solution, $x - x_0 = \frac{v_0}{k}(1 - \exp(-kt)) = \frac{v_0}{k}(1 - v/v_0)$, that is:

$$k(x - x_0) = v_0 - v$$

Actually, we could have gotten this much more easily. The trick is to use $\frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = \frac{dv}{dx} v$ and eliminate t from the EOM itself. If we do that, then the EOM becomes $\frac{dv}{dx} v = -kv$, i.e.

$$dv = -kdx$$

from which the above result $k(x - x_0) = v_0 - v$ can be derived quickly. As you can see, this trick is quite useful, when the question does not involve t .

This result can be described succinctly: **because the linear air resistance imposes a finite time scale $1/k$, the range of motion is limited to $\sim v_0/k$.** It just so happens that in this problem the \sim sign (“on the order of” or “up to a multiplicative constant”) can be replaced with the $=$ sign.

Checks: Two symbols for the initial condition: x_0 and v_0 . Notice that the dimension of v_0/k is $(L/T)T = L$, so our solution makes sense dimensionally, as it should. In the limit of $k \rightarrow 0$ we should recover $x = x_0 + v_0 t$. This is easily verified since $1 - \exp(-kt) \approx 1 - (1 - kt) = kt$ when $|kt| \ll 1$. In the opposite limit $k \rightarrow \infty$, we would expect that the particle will be instantly stuck. Indeed, we see that all terms vanish except x_0 , giving $x = x_0$.



Perturbation

Remember this word. It is of tremendous general importance. Here is what it means, for this problem. The above equation can be solved this way.

$$\begin{array}{ll}
 \dot{v} = -kv & \text{assume } -kv \text{ is small} \\
 \dot{v} \approx 0 & \text{0-th order equation; ignore the small term} \\
 v \approx v_0 & \text{0-th order solution} \\
 \dot{v} \approx -kv_0 & \text{1-st order equation; plug the 0-th order solution into the small term} \\
 v \approx v_0 - kv_0t & \text{1-st order solution} \\
 \dot{v} \approx -k(v_0 - kv_0t) & \text{2nd-order equation; plug the 1-st order solution into the small term} \\
 v \approx v_0 - v_0(kt) + \frac{1}{2}v_0(kt)^2 & \text{2nd-order solution} \\
 \dots &
 \end{array}$$

Here, the solution up to the **leading order correction** to v due to the air resistance is $v = v_0 - kv_0t$, the first order solution, since $-kv_0t$ is the primary effect of k on v if the perturbation assumption is correct, i.e. if k is indeed small. Integrating this, x can then be obtained up to *its* leading order correction. And consequently all other physical quantities (e.g., the time it takes for the particle to travel a certain distance) can be calculated up to the leading order correction (to $O(k)$, that is, for this case).

This problem is easy enough to see that this series will converge to, if you repeat this process indefinitely, exactly $v_0 \exp(-kt)$, which is not surprising at all.

The general importance of the perturbation method is that it is applicable to any equation, or any “normal”^a physics problem if a small parameter is present in the problem. For real life problems that are unsolvable by hand and involve too many degrees of freedom to be solvable by a computer in a finite time, the perturbation method is often the only tool that we have. In general, the first thing to do is to put your problem/equation in this form

$$F(x) = G(x)$$

where x is the unknown variable to solve for, $F(x) = \text{const.}$ is an easy-to-solve equation and $G(x)$ is a small term. Then, the perturbation method can be applied by solving $F(x) = 0$ (0-th order approximation) first, and then successively solving $F(x) = G(x_p)$, where x_p is the solution, not a variable, of the previous perturbation approximation. At each n -th order approximation step, the solution must contain expressions up to $O(\lambda^n)$, where λ is the “smallness” parameter, and only up to that order without any higher order terms. In principle, this method can be applied to any order to obtain any accuracy required.

^aWarning: Some problems of Nature are fundamentally “non-perturbative.” Those are also very interesting problems!
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2.2 Example 2.5 of text

Vertical motion, $F = -kmv$ plus gravity

The force is now $F = -mg - kmv$, and $v = \dot{z}$. $z \stackrel{def}{=} \text{vertical axis}$ (up is positive).

$$\begin{aligned}
 m\dot{v} &= -mg - kmv \\
 \frac{dv}{dt} &= -g - kv && \text{divide by } m, \text{ use Leibniz} \\
 \frac{dv}{g + kv} &= -dt && \times dt, /(g + kv) \\
 \frac{1}{k} \ln[(g + kv)/(g + kv_0)] &= -t && \text{integ. from } t=0 \text{ to } t=t \\
 \frac{g + kv}{g + kv_0} &= \exp(-kt) && \text{note, } kt = \text{dimensionless} \\
 \frac{v + g/k}{v_0 + g/k} &= \exp(-kt) && /k \text{ upstairs and downstairs, LHS} \\
 \frac{v + v_t}{v_0 + v_t} &= \exp(-kt) && v_t \stackrel{def}{=} g/k \\
 v &= (v_0 + v_t) \exp(-kt) - v_t && \times \text{ and } - \\
 v &= v_0 + (v_0 + v_t)[\exp(-kt) - 1] && + \text{ and } -
 \end{aligned}$$

Checks: The dimension of $v_t = g/k = (L/T^2)/(1/T) = L/T$ (velocity). (Known limit) If $k = 0$? (no air friction): $\exp(-kt) \approx 1 - kt$, we get $v = (v_0 + g/k)(1 - kt) - g/k = v_0 - gt$ (correct!). (Trend check) As t increases, $\exp(-kt)$ decreases, and so v decreases (becomes more negative). This is due to the pull of the gravity. However, due to the air resistance, v does not decrease indefinitely.

Terminal velocity As $t \rightarrow \infty$, we see that $v \rightarrow -v_t$. The terminal speed v_t is determined by the balance between the downward pull of the gravity and the upward resistance: $mg = mkv_t$.

Integrate v (2nd to last form) to get z :

$$z = z_0 + (v_0 + v_t)[1 - \exp(-kt)]/k - v_t t \quad (2.4)$$

Checks: Two symbols for the initial condition are z_0 and v_0 . You should challenge yourself to show that you get $z = z_0 + v_0 t - gt^2/2$ in the $k \rightarrow 0$ limit [Hint: use $\exp(-kt) \approx 1 - kt + (kt)^2/2$], and that $z(t \rightarrow \infty) = \text{const.} - v_t t$.



Galilean invariance in action

Notice that, compared to the previous example, the only thing that is different in this example is the finite terminal velocity, v_t . Prove to yourself the following. In the reference frame falling with velocity $-v_t$ (which is another inertial frame, due to Galilean invariance), the solutions of this example become exactly the same (with the symbol change $z \rightarrow x$) as those of the previous example.

2.3 Example 2.6 of text

No air resistance, Projectile motion, Initial speed v_0 , Angle θ

This is an elementary problem at the level of the introductory course. However, it doesn't hurt to review it. The key is that x, y motions can be tackled independently. Define the y axis to point up: gravity = $-mg\hat{y}$.

$$\begin{aligned}\ddot{x} &= 0 \\ \ddot{y} &= -g\end{aligned}$$

Integrate each ODE.

$$\begin{aligned}v_x &= v_{x,0} \\ v_y &= v_{y,0} - gt \\ x &= x_0 + v_{x,0}t \\ y &= y_0 + v_{y,0}t - \frac{1}{2}gt^2\end{aligned}$$

$v_{x,0} = v_0 \cos \theta \stackrel{\text{def}}{=} U$ (in the book) and $v_{y,0} = v_0 \sin \theta \stackrel{\text{def}}{=} V$ (in the book).

$$\begin{aligned}v_x &= v_0 \cos \theta \\ v_y &= v_0 \sin \theta - gt \\ x &= x_0 + v_0 t \cos \theta \\ y &= y_0 + v_0 t \sin \theta - \frac{1}{2}gt^2\end{aligned}$$

It immediately follows that the **trajectory**, the graph (x, y) , is a parabola, since x is a first order polynomial of t and y is a second order polynomial of t , and accordingly y is a second order polynomial of x .

$T \stackrel{def}{=} \mathbf{time\ of\ flight} \stackrel{def}{=} \mathbf{the\ time\ between\ the\ initial\ point\ to\ the\ final\ point.}$ Let us take the final point to be when the projectile comes back to the original height y_0 . Then, due to the obvious symmetry of the problem (see box), $T/2$ is the time when the projectile reaches the top of the trajectory.

$$T/2 = v_0 \sin \theta / g$$



Time-reversal symmetry (advanced)

What is the “symmetry” of this problem that we just mentioned as being “obvious”? It is called the **time reversal symmetry**. It sounds like fancy words, but it is not so difficult to define. Here is the definition. Say we have a specific Newton’s equation to solve. Pick an arbitrary motion satisfying that equation. Now, imagine playing the “movie” of the motion backwards. Is it another *possible* solution of the same Newton’s equation? If the answer is yes for any arbitrary solution, then the time-reversal symmetry is present for that Newton’s equation. Convince yourself that the current example has this symmetry, but the previous two examples do not. Whenever there is “dissipation” (friction, air-resistance) or a magnetic field, the time-reversal symmetry is not present, or, we say, is *broken*.

The **range of motion**, $R \stackrel{def}{=} x(t = T) - x_0$

$$\begin{aligned} R &= \frac{2v_0^2 \sin \theta \cos \theta}{g} && \leftarrow v_x T \\ &= \frac{v_0^2 \sin 2\theta}{g} \end{aligned}$$

This gives the familiar result that the maximum range is obtained if $\theta = 45^\circ$. This is the **optimum angle of throw** in the case when there is no air resistance.

2.4 Example 2.7 of text

Now, add air resistance $-km\vec{v}$.

The solution to this case consist of collecting all the ground work done in previous Sections (2.1, 2.2).

The x motion is that of an exponential approach to the finite range, $v_{x,0}/k$ as in Section 2.1. From Equation 2.2:

$$x = x_0 + \frac{v_{x,0}}{k}[1 - \exp(-kt)]$$

The y motion is that of Section 2.2. From Equation 2.4:

$$y = y_0 + (v_{y,0} + v_t)[1 - \exp(-kt)]/k - v_t t$$

where $v_t = g/k$.

When the air resistance is finite:

- The trajectory is no longer a parabola, as can be seen from the above two equations.
- In fact, the trajectory is no longer symmetric with respect to the point of maximum height. As v_x keeps decreasing, the trajectory becomes steeper as the mass comes down. (Think a fly ball in baseball.)
- The mechanical energy is not conserved, because it is lost at the rate $dE/dt = \vec{F}_{air} \cdot \vec{v} = -kmv^2$. Thus, when the particle is thrown up and then comes back to the same height, the speed will be smaller.
- The time-reversal symmetry is broken. This follows from the last two items.

Let us solve for the time of flight T for which $y = y_0$.

$$\begin{aligned} v_t T &= (v_{y,0} + v_t)[1 - \exp(-kT)]/k \\ \xi &= (1 + \alpha)[1 - \exp(-\xi)] \end{aligned} \quad \xi \stackrel{def}{=} kT, \quad \alpha \stackrel{def}{=} v_{y,0}/v_t = kv_{y,0}/g$$

In general, the solution of this equation for ξ can be easily obtained using Numerical methods.

It is very instructive, though, to consider the solution in limiting cases.

1. $k \rightarrow 0$. Low air resistance. The first approximation will be the value of ξ without any air resistance: $\xi \approx 2v_{y,0}k/g = 2\alpha$, and so the solution for ξ is a small number. Thus, we can expand

$$1 - \exp(-\xi) = \xi - \frac{1}{2}\xi^2 + \frac{1}{6}\xi^3 + \dots$$

So, we get

$$\xi = (1 + \alpha)\left(\xi - \frac{1}{2}\xi^2 + \frac{1}{6}\xi^3 + \dots\right)$$

Cancelling ξ , we get

$$1 = (1 + \alpha)\left(1 - \frac{1}{2}\xi + \frac{1}{6}\xi^2 + \dots\right)$$

Rearranging,

$$\frac{1 + \alpha}{2}\xi = \alpha + (1 + \alpha)\left(\frac{1}{6}\xi^2 + O(\xi^3)\right)$$

We get

$$\xi = \frac{2\alpha}{1 + \alpha} + 2\left(\frac{1}{6}\xi^2 + O(\xi^3)\right)$$

Now, **this is the form to which the perturbation method can be applied.** ($F(\xi) = \xi - \frac{2\alpha}{1+\alpha}$, and $G(\xi) = 2\left(\frac{1}{6}\xi^2 + O(\xi^3)\right)$ in the notation of that “Lisa box.”) Since ξ and α are of the same order, we expand the first term to second order as well

$$\frac{2\alpha}{1 + \alpha} = 2\alpha(1 - \alpha + \dots) = 2\alpha - 2\alpha^2 + \dots$$

Therefore,

$$\xi = 2\alpha - 2\alpha^2 + \frac{1}{3}\xi^2 + O(\alpha^3)$$

Thus, finally, plugging in $\xi \approx 2\alpha$, the 0-th order solution, on the RHS, we get

$$\xi \approx 2\alpha - \frac{2}{3}\alpha^2$$

Or,

$$T \approx T_0\left(1 - \frac{2}{3}\alpha\right)$$

where $T_0 = 2v_{y,0}/g$ is the unperturbed time of flight. So, the time of flight is decreased relative to the “un-perturbed” value, i.e. the value for $k = 0$.

2. $k \rightarrow \infty$. High air resistance. In this case, since $\xi \rightarrow \infty$ and $\alpha \rightarrow \infty$, and thus $\exp(-\xi)$ can be safely ignored, and $(1 + \alpha) \approx \alpha$. Thus, $\xi \approx \alpha$. As $\xi = kT$ and $\alpha = kv_{y,0}/g$, this means $T \approx v_{y,0}/g$. Namely,

$$T \approx T_0/2$$

In this case, T is determined by the downward motion only, to the first approximation. How is it so? Here is the explanation. The maximum height $\approx v_{y,0}/k$ (show it), and the time to reach the maximum height $\approx \ln(v_{y,0}k/g)/k$ (show it). On the way down, the terminal speed g/k is reached after time $\sim 1/k$, but the distance travelled during this time $\sim g/k^2 \ll v_{y,0}/k$, and the rest of the trip takes $\approx v_{y,0}/k/(g/k) \approx T$.

To summarize, even though the maximum height is near zero ($v_{y,0}/k$), because the terminal velocity is reached almost instantly in the downward motion, and because the terminal velocity is also very small (g/k), it takes a finite time for the downward motion ($(v_{y,0}/k)/(g/k) = v_{y,0}/g$), which is approximately equal to the total time of flight.



$k \rightarrow 0$ or $k \rightarrow \infty$, that is confusing!

If you carefully examine the above discussions, you will note that $k \rightarrow 0$ really means $\alpha = v_{y,0}/v_t \ll 1$ and $k \rightarrow \infty$ really means $\alpha \gg 1$. For example, even if k is large, say 10 sec^{-1} , if $v_{y,0}$ is very small, say 0.1 m/s , then this corresponds to a very small air resistance problem, that is, case 1 not case 2. **When a certain parameter is said to be small or large, always look for a dimensionless version of that parameter in the equation. It is that dimensionless parameter which is large or small.** In the current example, α is the dimensionless version of k . This makes sense. When a certain parameter is said to be large or small, that sentence by itself has no meaning. What does it mean that $k = 10 \text{ sec}^{-1}$ is large or small? The unit can be changed, and the number can look large or small in an arbitrary way, e.g. $k = 0.01 \text{ millisecc}^{-1}$. The only way to judge whether k is large or small is to compare it with another time constant of the problem, which is, in our current example, $g/v_{y,0}$. In this way, $\alpha = k/(g/v_{y,0})$, can be also understood as the ratio of the time constant for the air resistance and the time constant of the free fall.

From these considerations of the two limiting cases, one may expect that T gradually decreases from T_0 to $2T_0$, as k increases from 0 to ∞ . That this is indeed the case can be verified both numerically and analytically (left for your optional work).

Now, let us ask a question. Will the optimum throw angle be different from $\theta = 45^\circ$? If so, will it be greater than 45° or less than it?

The qualitative answer is easy to get, given the above results.

Let us first analyze, in the case of no air resistance, what determined the optimum throw angle. Two factors.

1. The longer the time of flight, the better.
2. The larger $v_{x,0}$, the better.

These are conflicting requirements, the first $\propto v_{y,0} \propto \sin \theta$ and the second $\propto \cos \theta$. Since the range is the product of the time of flight and $v_{x,0}$, the answer is to optimize the product $\sin \theta \cos \theta$.

Now, consider air resistance. Notice that the time of flight is still $\propto v_{y,0}$ and the range of motion is still $\propto v_{x,0}$. However, the range of motion is not directly proportional to the time of flight any more. Essentially, the x motion crawls to a stop, after the time scale $1/k$ has passed. So, there is a sort of “diminishing return” for optimizing the time of flight. To make up for the diminishing return, it is necessary to increase $v_{x,0}$, in order to move the projectile farther in the initial stage. Thus, the answer is that $\theta < 45^\circ$ is the optimum throw angle.

While this qualitative argument is precise, it does not answer the question “how much?”. One can verify this argument and get a quantitative answer analytically by examining the two limits considered above (homework).

2.5 Example 2.10 of text

Motion of a charge in a uniform \vec{B} field.

$\vec{B} = B_0 \hat{z}$. (This is a more common convention. The textbook chooses the y direction.) The Lorentz force $\vec{F} = q\vec{v} \times \vec{B}$, where q is the electric charge of the particle.

(Also, below, I prefer using $\hat{x}, \hat{y}, \hat{z}$ instead of $\vec{i}, \vec{j}, \vec{z}$, as i, j, k are commonly used as summation indices!)

Any vector product can be calculated, if the following basic rules are noted, along with other usual properties of multiplication (associative rules, distributive rules, ...).

$$\begin{aligned}\vec{A} \times \vec{B} &= -\vec{B} \times \vec{A} && \text{anti-commutative!} \\ \hat{x} \times \hat{y} &= \hat{z} \\ \hat{y} \times \hat{z} &= \hat{x} \\ \hat{z} \times \hat{x} &= \hat{y}\end{aligned}$$

All of the following properties can be derived from the above. You should always know these as well.

1. $\vec{A} \times \vec{A} = 0$.
2. **Right screw/hand rule** applies. Rotate \vec{A} towards \vec{B} (involving the shortest angular displacement possible). How would a right-handed screw move along its axis on such a rotation? That is the direction of $\vec{A} \times \vec{B}$.
3. $\vec{A} \times \vec{B}$ is perpendicular to both \vec{A} and \vec{B} : $(\vec{A} \times \vec{B}) \cdot \vec{A} = (\vec{A} \times \vec{B}) \cdot \vec{B} = 0$.
4. $|\vec{A} \times \vec{B}|$ is the **area of the parallelogram** spanned by \vec{A} and \vec{B} , i.e. twice the area of the triangle formed by \vec{A} and \vec{B} . It is $AB \sin \theta$.
5. $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \sum_{i,j,k=1}^3 \varepsilon_{ijk} \hat{i} A_j B_k$, where ε_{ijk} is the Levi-Civita symbol ($\varepsilon_{123} = 1$ and it changes sign whenever two indices are swapped, i.e. permuted), and $\hat{1} \stackrel{def}{=} \hat{x}$, $\hat{2} \stackrel{def}{=} \hat{y}$, $\hat{3} \stackrel{def}{=} \hat{z}$. Here, the middle term means the determinant of the 3×3 matrix. It is a *fake* matrix, constructed for mnemonics only, where $\hat{x}, \hat{y}, \hat{z}$ are to be thought as "just numbers."
6. In other words, $(\vec{A} \times \vec{B})_3 = A_1 B_2 - A_2 B_1$ and, by **cyclic permutations of indices**, $(\vec{A} \times \vec{B})_2 = A_3 B_1 - A_1 B_3$ and $(\vec{A} \times \vec{B})_1 = A_2 B_3 - A_3 B_2$.
7. $(\vec{A} \times \vec{B}) \cdot \vec{C} = (\vec{B} \times \vec{C}) \cdot \vec{A} = (\vec{C} \times \vec{A}) \cdot \vec{B} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$. The magnitude $|(\vec{A} \times \vec{B}) \cdot \vec{C}|$ is the volume of the parallelepiped formed by these three vectors.
8. Finally, you cannot live without the following identity, when studying the vector product! $\sum_{k=1}^3 \varepsilon_{ijk} \varepsilon_{lmk} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$.

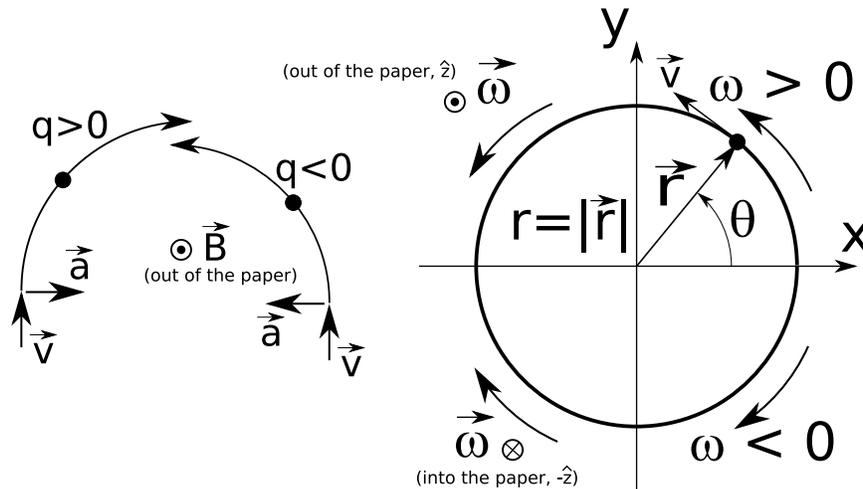
2.5. EXAMPLE 2.10 OF TEXT

The Lorentz force is then $q\vec{v} \times B_0\hat{z} = q(v_yB_0\hat{x} - v_xB_0\hat{y})$. Notice that there is no force along the z direction, since $\hat{z} \times \hat{z} = 0$. So, $v_z = v_{z,0}$.

Let us use the inertial frame, which is moving at $v_{z,0}\hat{z}$. Then, there is no z motion, and we can simply deal with x and y motions only.

$$\begin{aligned} ma_x &= qv_yB_0 \\ ma_y &= -qv_xB_0 \\ a_x &= -\omega_c v_y & \omega_c &\stackrel{def}{=} -qB_0/m \\ a_y &= \omega_c v_x \\ \vec{a} &= \omega_c(-v_y\hat{x} + v_x\hat{y}) \\ \vec{a} \cdot \vec{v} &= 0 & \omega_c(-v_yv_x + v_xv_y) &= 0 \end{aligned}$$

Here, $\omega_c = -qB_0/m$ has the dimension of inverse time, and it is call the “**cy-clotron frequency**.” The motion is a **uniform circular motion**, with the angular frequency given by ω_c . How do we know this? (1) The acceleration is always perpendicular to the velocity. So, it is a centripetal acceleration. So, it is a circular motion. There is no tangential acceleration, so v will be constant. So, a uniform circular motion. (2) The magnitude of the acceleration is, $a = \sqrt{\omega_c^2v_x^2 + \omega_c^2v_y^2} = |\omega_c|v$. This is precisely the uniform circular motion relation that you learned in an elementary course, with the angular frequency $|\omega_c| = 2\pi/T$ where T is the period.²



These arguments are quite satisfactory, but let us do some more work here, now that we know precisely what it means to solve a Newton’s equation. Notice that we

²Unfortunately, T is a versatile symbol, used for multiple purposes. Sometimes it is “time” (dimension). Sometimes it is period. Sometimes it is kinetic energy. Or, temperature. Usually the context makes the meaning unambiguous.

have a 2D problem of one particle, so the number of integration constants is four. How many numbers do we need to characterize a uniform circular motion? 2 for the position of the center. 1 for the radius r . 1 for the speed v . 1 for the phase (θ_0) ,³ the initial angular position. It seems that we have 5 – which is too many. But, remember that $v = 2\pi r/T$, and so r and v are physically constrained by ω_c , i.e. by B_0 and q/m . So we have 4 numbers. Good. As the origin of the coordinate system is arbitrary, let us choose the most convenient coordinate system so that the center of the circle is the origin. Then, we expect to have two integration constants in the solution.

Now, we are ready to write down the solution. By using the above coordinate system, and identifying $\theta = \omega_c t + \theta_0$, we get

$$\begin{aligned}x &= r \cos(\omega_c t + \theta_0) \\y &= r \sin(\omega_c t + \theta_0)\end{aligned}$$

Here, r, θ_0 are two integration constants that need to be fixed by the initial condition. Upon taking the derivative, $\dot{x} = -\omega_c y$, and $\dot{y} = \omega_c x$. Repeating, $\ddot{x} = -\omega_c \dot{y}$, and $\ddot{y} = \omega_c \dot{x}$. This agrees with the equations that we wrote above ($a_x = -\omega_c v_y$ and $a_y = \omega_c v_x$). This ends the proof that the above solution is the general solution.

If we include the z motion, then the motion is that of a cylindrical spiral motion, a circular motion in the $x - y$ plane plus a uniform translation along the z axis.

Notice that we defined ω_c so that it is negative for the positive charge and positive for the negative charge. This is the result of our setting up the coordinate system as shown above.

Note that the angular velocity vector $\vec{\omega}$ is in general defined through a right-hand rule. We will study more details later, but, at this point, it should suffice to study the direction of $\vec{\omega}$ in the above diagram.

In the next lecture, we will show how to deal with this type of equation from a more mathematical point of view.

As you can see, the cyclotron motion can be useful to figure out what the sign of the particle's electric charge is, or what the kinetic energy of the particle is after a collision, if q/m is known.

Lastly, note that the time reversal symmetry is broken for this problem. However, different from dissipative problems (friction, air resistance), which also break the time reversal symmetry, the mechanical energy is conserved here. The Lorentz force is perpendicular to the velocity, and so it actually **does not do any work**.

³ Actually, “phase” is defined as θ . And so, θ_0 should be called the *initial* phase, strictly speaking.