Notes for Lecture 1

Newton's laws

In general, my notes are not complete. You should read both my notes and the textbook.

Chapter 1 materials will be spread around and parts of them will be taken up during my lectures. We do that in this lecture, for example. In general, I will try to explain all the core mathematics as much as it is necessary to do so and as much as we have time for it. However, many basic things will be assumed to be known, and I will simply remind you of them. For instance, the meaning of the gradient vector should be familiar from calculus. If you are weak in basic math, I would recommend that you go through Chapter 1 carefully now.

1.1 Particle, body, degrees of freedom, dimension

Paricle, Body The "particle" is a fundamental concept in classical mechanics. By "particle" we mean an object whose size can be ignored. It is a mathematical point, assigned with a mass value. This is an approximation, of course, and what we call "particle" can vary greatly in size. For instance, when we consider the motion of the Earth around the Sun, then we might call the Earth a "particle." Or a "body," since we are talking about a celestial body. But, when we consider an apple falling from a tree, then the Earth is definitely not a particle, but we might call the apple a "particle" to a good approximation. But, when the apple hits the ground and breaks into hundreds of pieces, then . . . . In any case, note that "particle" in classical mechanics should not be confused with "fundamental particle" such as the electron, the neutrino, the proton etc. The mechanical law governing these fundamental particles is quantum mechanics, not classical mechanics. In general, a particle in classical mechanics is a composite object consisting of a very large number of fundamental particles. Indeed, Newton's law should be thought of as an emerging law when a large number of quantum particles are coalesced together. As such any "particle" in classical mechanics have a large internal degrees of freedom (see below). When the internal degrees of freedom are important, the object can no longer be considered as a point, and the term "body" would be much more appropriate. Even then, when we refer to the average motion of the body, we may still use the term "particle."

Dimension The spatial dimension is the number of coordinates to specify the position of one particle. In classical mechanics, time is just a parameter, and so the spatial dimension is the only dimension that we care about. The spatial dimension of our world is 3. Consider an airplane flying in the sky, we need 3 coordinates. These could be the Cartesian coordinates, (x, y, z), the spherical coordinates, (r, θ, ϕ), the cylindrical coordinates (ρ, ϕ, z), or most likely in real life, (longitude, latitude, altitude). Whatever coordinate system we use, the number of coordinates is 3. Thus, the (spatial) dimension is 3. This is true for any motion. However, if a particle is constrained to go through a linear motion only, then effectively we need only one coordinate, say x, to specify its position. Thus, the effective dimension is 1. I will use the symbol, D, for dimension. $D = 3$ for 3 dimensions, and $D = 1$ for 1 dimension, etc. Also, I will use the short-hand 1D, 2D, or 3D, to mean one-dimensional, two-dimensional, or three-dimensional, respectively.

Degrees of freedom For a given mechanical system, the degrees of freedom refers to the number of coordinates necessary to specify the positions of all particles. For one particle system, the degrees of freedom is thus equal to D. For a many particle system consisting of N particles, the degrees of freedom is $N \times D$.

Dimension A more general concept than the spatial dimension is the dimension of a physical quantity. Recall that the seven base units of the SI unit system are m, kg, s, A, K, cd, and mol. For mechanical problems, we are concerned with the first three only. Let us say that a mechanical quantity has the SI unit $\mathrm{m}^\alpha\mathrm{kg}^\beta\mathrm{s}^\gamma$. Then, the (physical) dimension of that quantity is defined as $L^\alpha M^\beta T^\gamma$, where L means length, M means mass, and T means time. For example, the angular frequency ω has the dimension T^{-1}. We say that it has the dimension of inverse time. The energy has the dimension $L^2 M T^{-2}$. In thermal physics, we learn that the heat has the same dimension as the energy. **No two physical quantities can be equal to each other, if their dimensions are different.** For this reason, if you solve a problem using symbols, then the first thing that you must check is the dimension. This is because, if the dimension is incorrect, then there must have been a mistake that you need to correct.

How to avoid getting things wrong...

As you undoubtedly know by now, you should do almost all physics problems **symbolically**. That is, you should use symbols for variables, obtain a symbolic expression for your answer first, before finally plugging in numbers to get a numerical answer, if required. Next, you should double check your answer. The better you are, the more you know about your weakness as well as your strength, and double checking should be pretty much an instinctive routine while you are doing problems. Here, I offer some guidelines how to **double check your answers**, and be sure about your answer, before anybody makes that potentially unpleasant judgement about your answer. You should apply these guidelines to your symbolic answer first and foremost, if applicable, and then to your numeric answer. These guidelines are mighty important to prevent any potential embarrassment, not to mention a deep negative impact on your scores. The potential negative impact that you will be avoiding by heeding these rules can be immense for the top item of the list, and goes down gradually in its severity as the list goes down.

- Does the **physical dimension** of my answer make sense?
- Does the **sign** or **(scaling) trend** make sense?
- Does the answer make sense in **known limits**, if any?
- Does the **order of magnitude** of my answer make sense?
- D'oh! Did I drop 2, π, ... somewhere?

1.2 Vectors and matrices

Solving a mechanical problem usually requires setting up a coordinate system. In doing so, we are free to choose a coordinate system that is the most useful and the most elegant *for us*. **The eventual physics answer is independent of the coordinate system**, so it does not matter what coordinate system we use. The coordinate system is something that we draw in space, out of the blue, just to make it easy to calculate things. It is an essential device for us, but physics does not, and should not, depend on our choice of the coordinate system.

Consider the Cartesian coordinate system. The coordinates consist of D numbers, and in three dimensions they are written as x, y, z. The position corresponding to these coordinates are usually denoted with a symbol such as $\vec{r}$ or $\vec{x}$. For the reason that will become clear below, we will write coordinates **column-wise**, as in

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \text{if } D = 3 \tag{1.1}$$

Now, imagine rotating the coordinate system, or reflecting the coordinate system (like reflecting the world in a mirror), or inverting the coordinate system (reflecting all coordinates), while, in all cases, the origin of the coordinate system remains fixed. These are examples of **coordinate transformations**. We consider the particle position as fixed, as we instantaneously make the coordinate transformation. Namely, physics is one and the same, but our description can be different depending on the coordinate system. In the new coordinate system, the same position is now represented by different numbers, x', y', z'.

$$\vec{r}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \qquad \text{if } D = 3 \tag{1.2}$$

Coordinate transformations considered above, namely rotation, reflection, and inversion, are **orthogonal transformations**.

Coordinate transformation $T : \vec{r} \to \vec{r}' = T(\vec{r})$

In general, $T(\vec{r})$ can be any function.

Linear transformation $L : \vec{r} \to \vec{r}' = L(\vec{r}) = \overleftrightarrow{L}\vec{r}$

Here, $\overleftrightarrow{L}$ is a "**square matrix**." In general, a **matrix** means a rectangular array of numbers.

Any coordinate transformation that displaces the origin is an example of a non-linear transformation.

Orthogonal transformation $O : \vec{r} \to \vec{r}' = \overleftrightarrow{O}\vec{r}$

Here, $\overleftrightarrow{O}$ is an **orthogonal matrix**, which can be defined as a square matrix satisfying any one of the following four equivalent properties. (1) The column vectors of $\overleftrightarrow{O}$ are orthonormal. That is, each column vector is a unit length vector, and perpendicular to one another. (2) So are the row vectors of $\overleftrightarrow{O}$. (3) $\overleftrightarrow{O}\overleftrightarrow{O}^t = \overleftrightarrow{O}^t\overleftrightarrow{O} = \overleftrightarrow{1}$ (t = transpose). (4) $\overleftrightarrow{O}^{-1} = \overleftrightarrow{O}^t$.

O_{ij} corresponds to λ_{ij} of Chapter 1 of the book.

So, an orthogonal transformation is a special kind of linear transformation. Also, note that expressions such as $\overleftrightarrow{L}\vec{r}$ and $\overleftrightarrow{O}\vec{r}$ make sense as matrix multiplication, only if $\vec{r}$ is a *column* vector, which is our convention here. Finally, note that I am using a bi-directional arrow over a symbol to mean a matrix quantity, as in Gibbs' "dyadic notation." In particular, $\overleftrightarrow{1}$ means the identity matrix, whose diagonal elements are all 1's and whose non-diagonal elements are all 0's. A quick and dirty but very often used short-hand for $\overleftrightarrow{1}$ is 1. I.e., if you see an expression such as $\overleftrightarrow{O}\overleftrightarrow{O}^t = 1$, you should automatically upgrade 1 on the right hand side to an identity matrix of correct dimensions. This applies not only to 1, but also to any number, which, if equated with a matrix, should be interpreted as that number times the identity matrix.

It is worth noting the **fundamental definition of a linear transformation**:

$$L(a\vec{r}_1 + b\vec{r}_2) = aL(\vec{r}_1) + bL(\vec{r}_2) \tag{1.3}$$

for any positions $\vec{r}_1$ and $\vec{r}_2$ and any numbers a, b. This definition is completely equivalent to the above definition: L is a linear transformation if L can be represented by a matrix multiplication $L(\vec{r}) = \overleftrightarrow{L}\vec{r}$.

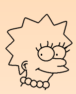
Just like the position, any physical quantity can be represented as a set of numbers, given a coordinate system. Let's take an arbitrary physical quantity. We measure it in one coordinate system ($\vec{r}$), and call it Q. The same quantity can be measured in the transformed coordinate system ($\vec{r}'$), and we will call it Q'. How the transformation from Q to Q' is related to, or not related to, the coordinate transformation itself, is an important characteristic of the physical quantity. For one, that is how we define a vector quantity and a scalar quantity. The following is a more precise definition than a vague definition that one encounters in elementary physics courses.

Vector Any physical quantity whose representation, $\vec{V}$, transforms just like the position $\vec{r}$, for an arbitrary linear coordinate transformation, is called a vector quantity. Namely, $\overleftrightarrow{L}\vec{V} = \vec{V}'$.

Examples of vector quantities include position (by definition!), velocity, momentum, force, angular momentum, and acceleration.

Scalar Any physical quantity whose representation, S, remains unchanged by an arbitrary linear coordinate transformation is called a scalar quantity. Namely, $S = S'$.

For instance, time is independent of coordinate systems in classical mechanics, and thus it is a scalar quantity.[1] Mass is another example. For given vector quantities, scalar quantities can be derived from them also. E.g., the magnitude of a vector and the angle between two vectors are scalar quantities, according to the following property.



Scalar product is scalar, indeed.

The **scalar product** of two vectors, $\vec{A}$ and $\vec{B}$, is defined as

$$\vec{A} \cdot \vec{B} = \sum_i A_i B_i$$

where A_i's (B_i's) are the Cartesian components of $\vec{A}$ ($\vec{B}$). It is invariant under an orthogonal transformation. In other words, the scalar product is indeed **a scalar quantity for orthogonal transformations**.

[1]In the relativistic mechanics of Einstein, time is no longer absolute when speeds close to the speed of light are involved. Instead, time should be considered as another axis of the coordinate system, part of the four dimensional "space-time" vector. It is no longer a scalar.

We will see how this property arises, in a little bit, but let us discuss some key facts, first.

The scalar product of two vectors can also be understood as a matrix multiplication (recall that a vector is also a matrix after all).

$$\begin{aligned}\vec{A}\cdot\vec{B} &= \vec{A}^t\vec{B}\\ &= \vec{B}^t\vec{A}\end{aligned} \tag{1.4}$$

On the right-hand side, the matrix multiplication of a row vector ($\vec{A}^t$ or $\vec{B}^t$) and a column vector ($\vec{B}$ or $\vec{A}$, respectively) is seen to result in a number.

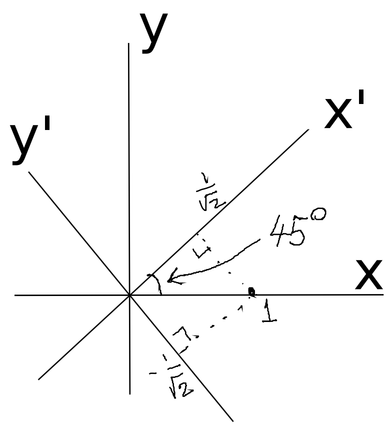
Scalar product has other names: **dot product** and **inner product**. Be careful not to change the order of a matrix product, since that is generally not allowed! In the current case, definitely $\vec{A}^t\vec{B} \neq \vec{B}\vec{A}^t$. Actually, $\vec{B}\vec{A}^t$ results in a square matrix of dimensions $D \times D$! This is the so-called the **outer product** of two vectors. Finally, a third kind of vector product is possible: this is the **vector product** or the **cross product**: $\vec{A}\times\vec{B}$. We will discuss the vector product later, when we need to. Different from other products, the vector product is defined only in 3D or 7D.

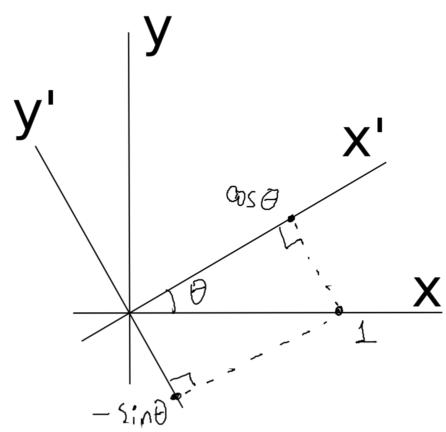
Often, a description of a linear coordinate transformation is given, and you need to construct the transformation matrix fast. Here is how you do it.

How to construct a transformation matrix fast?

- Figure out how old unit vectors, $\hat{x}$ first, then $\hat{y}$ and so on, are represented in the new coordinate system.
- Write your answers as column vectors from left to right.
- Voilà, you have the transformation matrix.

For example, consider rotating a 2D coordinate system by 45° (the left figure). In this case, $\begin{pmatrix}1\\0\end{pmatrix} \to \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$, and $\begin{pmatrix}0\\1\end{pmatrix} \to \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$. Therefore, the transformation matrix is given by $\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\-1&1\end{pmatrix}$. In general, rotating a 2D coordinate system by θ:





$$\overleftrightarrow{O} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{1.5}$$

In terms of $\vec{o}_1 = \left(\begin{smallmatrix} \cos\theta \\ -\sin\theta \end{smallmatrix}\right) = \overleftrightarrow{O}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\vec{o}_2 = \left(\begin{smallmatrix} \sin\theta \\ \cos\theta \end{smallmatrix}\right) = \overleftrightarrow{O}\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, we can re-write this in a more general form with a short-hand notation[2]

$$\overleftrightarrow{O} = \begin{pmatrix} \vec{o}_1 & \vec{o}_2 \end{pmatrix} \tag{1.6}$$

Using this form, we can discuss the general properties of an orthogonal matrix. Now, the fundamental definition of an orthogonal transformation is that the column vectors are orthonormal to each other, $\vec{o}_i^{\,t}\vec{o}_j = \delta_{i,j}$, where $\delta_{i,j}$ is the **Kronecker-delta symbol** (1 if $i = j$ and 0 otherwise). If this is the case, then

$$\overleftrightarrow{O}^t\overleftrightarrow{O} = \begin{pmatrix} \vec{o}_1^{\,t} \\ \vec{o}_2^{\,t} \end{pmatrix} \begin{pmatrix} \vec{o}_1 & \vec{o}_2 \end{pmatrix} = \begin{pmatrix} \vec{o}_1^{\,t}\vec{o}_1 & \vec{o}_1^{\,t}\vec{o}_2 \\ \vec{o}_2^{\,t}\vec{o}_1 & \vec{o}_2^{\,t}\vec{o}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This proves that the inverse of an orthogonal matrix is its transpose ($\overleftrightarrow{O}^{-1} = \overleftrightarrow{O}^t$), from which all the rest of the properties of an orthogonal matrix, as listed a few pages back, follow.

What we just showed generalizes to an orthogonal matrix in any dimensions.

[2] Note that in this new notation we are repeating the brackets () for the column vectors and then for the matrix. This is no problem. Those brackets are just visual aids, and not an integral part of the definition of a matrix. A matrix is simply a rectangular array of numbers.

Now, let us consider two vectors $\vec{A}$ and $\vec{B}$. Under an orthogonal transformation, $\vec{A}' = \overleftrightarrow{O}\vec{A}$ and $\vec{B}' = \overleftrightarrow{O}\vec{B}$. The scalar product

$$\begin{aligned}
\vec{A}' \cdot \vec{B}' &= (\overleftrightarrow{O}\vec{A})^t \overleftrightarrow{O}\vec{B} && (\because \vec{A} \cdot \vec{B} = \vec{A}^t \vec{B}) \\
&= \vec{A}^t \overleftrightarrow{O}^t \overleftrightarrow{O}\vec{B} && (\text{matrix transpose rule } (MN)^t = N^t M^t) \\
&= \vec{A}^t \overleftrightarrow{1}\vec{B} && (\because \overleftrightarrow{O} \text{ is an orthogonal matrix.}) \\
&= \vec{A}^t \vec{B} \\
&= \vec{A} \cdot \vec{B}
\end{aligned}$$

This shows that the scalar product is invariant under any orthogonal transformation. So is the **magnitude of a vector**, since $A \overset{def}{=} |\vec{A}| \overset{def}{=} \sqrt{\vec{A} \cdot \vec{A}}$, and the **angle between two vectors**, $\angle(\vec{A}, \vec{B}) \overset{def}{=} \cos^{-1}\left(\frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}\right)$.

What kind of transformations are linear, but not orthogonal? Stretching or skewing (shearing), with the origin fixed.

1.3 Newton's laws

Velocity vector

$$\vec{v} \overset{def}{=} \frac{d\vec{r}}{dt} \overset{def}{=} \dot{\vec{r}}$$

Here, we are introduced to the concept of the derivative of a function ($\vec{r}(t)$) with respect to its variable (time t). The first notation is due to Leibniz, who discovered calculus about the same time as Newton. In the second form, the dot on function is Newton's short-hand for the time derivative. Newton's notation is compact, but the Leibniz notation is arguably richer. The chain rule, such as

$$\frac{df}{dt} = \frac{df}{dg}\frac{dg}{dt}$$

is very natural in Leibniz notation, as dg just "cancels out." Also, one often writes

$$\frac{df}{dt} = \frac{d}{dt}f = \left(\frac{d}{dt}\right) f$$

where $\left(\frac{d}{dt}\right)$ is now easily recognized as an **operator** on a function. Moreover, it is a **linear operator** since

$$\frac{d}{dt}(af(t) + bg(t)) = a\frac{d}{dt}f + b\frac{d}{dt}g$$

for any constants a, b and any functions f, g. Compare this definition with the fundamental definition of a linear coordinate transformation of the previous section (Equation 1.3 in page 5), and note the similarity.

Acceleration vector

$$\vec{a} \stackrel{def}{=} \frac{d\vec{v}}{dt} = \dot{\vec{v}} = \ddot{\vec{r}} = \left(\frac{d}{dt}\right)^2 \vec{r}$$

Note that here, the **second-derivative operator** $\left(\frac{d}{dt}\right)^2$ is also a **linear operator**.

Newton's first law A particle with a constant velocity will remain at that velocity if not acted upon by a force. (Law of inertia)

Newton's second law A particle acted upon by a force, $\vec{F}$, will change its motion, according to

$$\vec{F} = \frac{d\vec{p}}{dt} \tag{1.7}$$

where

$$\vec{p} = m\vec{v}$$

is the **momentum** of the particle. If the mass is constant, then

$$\vec{F} = m\vec{a}. \tag{1.8}$$

Newton's third law If particle 1 exerts a force $\vec{F}_{21}$ on particle 2, then particle 2 exerts an equal and opposite force $\vec{F}_{12} = -\vec{F}_{21}$ on particle 1. ("Action-reaction")

What do Newton's laws really mean? We can take it to mean that the first two laws are the definition of "force," while the "mass" is taken as a known quantity (i.e. measured with a scale – the principle equivalence would guarantee that the **gravitational mass** – the mass measured as the weight divided by $\vec{g}$ – is the same as the **inertial mass** – the mass that enters in Newton's 2nd law). Then, really the third law is the only physical law. A different view is also possible (see discussion in the textbook).

Notice that the third law is equivalent to the **conservation of momentum** in the following sense. Consider two particles as an isolated system. Then, the total force acting on this system is $\vec{F}_t = \vec{F}_{21} + \vec{F}_{12}$. The total momentum, $\vec{P} = \vec{p}_1 + \vec{p}_2$, will then change according to the 2nd law: $\dot{\vec{P}} = \vec{F}$. So, one can see that $\vec{P}$ = constant (conserved) if and only if $\vec{F} = 0$, i.e. $\vec{F}_{21} = -\vec{F}_{12}$.

Newton's 3rd law fails?!... (optional reading)

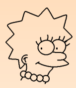
You may hear or read that Newton's third law fails (or "appears to fail") in certain situations (e.g. when two moving charges interact in a certain geometry – do your Google search on this, if you are curious). It would seem that when such an example is discussed, a presumably classical electro-magnetic problem is invariably imagined. But some thoughts given to such a "classical" problem would convince any reasonable person that it would be quite impossible to consider such an experiment without worrying about other many body effects, such as screening and polarization, that must happen for classical systems. In order to help any of you, who might be interested in this type of discussion, here I will state some guiding principles to keep in mind.

1. The momentum conservation is a much more general principle than Newton's 3rd law.
2. If the momentum is conserved and if any classical system can be divided into two classical particles exchanging forces, then Newton's 3rd law *must* follow, as we just showed.
3. If the momentum is conserved and if 2 fails, then it follows that the assumption of the "two separate classical particles" is invalid. Instead, quantum mechanics should be used. Newton's laws are meaningless to discuss.

Please read my note at the link "Nature-Laws" on the course web site for some more related discussion. Also, read Feynman, vol. II, chapter 28.

Inertial reference frame A reference frame is another name for the coordinate system. A reference frame in which Newton's laws are valid is called an inertial reference frame.

I would invite you to think about what this definition really means. You should think of experiments you did in your laboratory course, and how you could verify Newton's laws. It is important to realize that there is no such thing as an absolute inertial reference frame.



Galilean invariance

Suppose there is an inertial reference frame, and another reference frame that is in uniform motion relative to that. Then, the second reference frame is also an inertial reference frame. This is the **relativity principle** at the classical mechanics level. Roughly put, the relativity principle means that physics, or physical laws, are independent of reference frames. Oohoo!

Since the origin of the second reference frame is moving at a constant velocity relative to the first, $\vec{r}' = \vec{r} + \vec{V}t + \vec{R}_0$ where $\vec{R}_0$ is a constant position vector and $\vec{V}$ is a constant velocity vector. Note that $\ddot{\vec{r}}' = \ddot{\vec{r}}$. Thus, Newton's 2nd law in the first frame, $\vec{F} = m\ddot{\vec{r}}$ means $\vec{F} = m\ddot{\vec{r}}'$.

Free body diagram Please study section 2.4 and examples 2.1, 2.2 and 2.3, carefully, if you are not sure about this. I am not covering this basic topic, in the interest of time. We will be practicing free body diagrams later though. If you know that you are weak on this topic, you should definitely strengthen your knowledge on this now by reading section 2.4.

1.4 Solving Newtonian equation of motion

For a one body problem with a constant mass, we need to solve

$$\vec{F} = m\vec{a} = m\ddot{\vec{r}}.$$

Let us consider a 1D problem first.

$$F = m\ddot{x}.$$

This is a so-called **second order differential equation** for $x(t)$, since it involves a second derivative but no higher derivative.

$F = F(x, \dot{x}, t)$, as far as we know. I.e., F is an explicit function of position, velocity and time, but no higher order derivatives of x.

This makes the above equation completely solvable. How? Like the computer does. There is one condition, though. We need to know x and $\dot{x}$ at the initial time (**initial condition**). Say, $x_0 = x(t = 0), v_0 = \dot{x}(t = 0)$. Then, $x(t)$ at any time is known! (**Newtonian determinism**)

Here is how it goes.

$$
\begin{aligned}
x_0, v_0 &\xrightarrow{\text{determine}} F_0 \stackrel{def}{=} F(x_0, v_0, t = 0) \\
F_0 &\xrightarrow{\text{gives}} a_0 = F_0/m \\
x_0, v_0 &\xrightarrow{\text{determine}} x_\epsilon \stackrel{def}{=} x(t = \epsilon) = x_0 + v_0\epsilon \\
v_0, a_0 &\xrightarrow{\text{determine}} v_\epsilon \stackrel{def}{=} v(t = \epsilon) = v_0 + a_0\epsilon \\
x_\epsilon, v_\epsilon &\xrightarrow{\text{gives}} F_\epsilon \stackrel{def}{=} F(x_\epsilon, v_\epsilon, t = \epsilon) \\
x_\epsilon, v_\epsilon, F_\epsilon &\xrightarrow{\text{repeat}} x_{2\epsilon}, v_{2\epsilon}, F_{2\epsilon} \\
x_{2\epsilon}, v_{2\epsilon}, F_{2\epsilon} &\xrightarrow{\text{repeat}} x_{3\epsilon}, v_{3\epsilon}, F_{3\epsilon} \\
&\xrightarrow{\text{and repeat}} \\
&\ldots \\
&\xrightarrow{\text{we can get}} x(t), v(t) \text{ for } \textit{any} \text{ time } t!
\end{aligned}
$$

This basically summarizes the concept of calculus, of course. Divide the finite time interval into many a pieces of very very short of intervals, where the linear approximation is practically precise, and put all pieces together. ϵ is the infinitesimal, which means a very small finite number, for which the Taylor series approximations $x(t = \epsilon) \approx x_0 + v_0\epsilon$ and $v(t = \epsilon) \approx v_0 + a_0\epsilon$ become *identities*, as used above, within the error bars of the finest measurement that we can make.

Why are the two constants, x_0 and v_0, necessary? Because we are doing a double integration, due to the differential equation being a second order equation. So, two constants are necessary for the most general solution. We say that there are **two integration constants** in solving a motion of a particle in one dimension. These constants may be fixed by any two independent physical conditions given in an actual problem.

The above procedure is readily generalized to higher dimensions as well. All we need to do is to put a vector sign on symbols, x, v, and F.

$$\vec{F}(\vec{x}, \vec{v}, t) = m\vec{a}.$$

The key steps are

$$\vec{v}(t = \epsilon) = \vec{x}_0 + \vec{v}_0\epsilon$$
$$\vec{x}(t = \epsilon) = \vec{v}_0 + \vec{a}_0\epsilon$$

In this case, there will be **precisely** $2D$ **integration constants**, $\vec{x}_0$ and $\vec{v}_0$.

Seeing that the Newton's second law equation of motion can be solvable for any given force is excellent. This type of solution, if obtained with the help of a computer, is called a **numerical solution**. For some important classes of problems, we can find the solution with paper, pencil and some math tricks. Such a solution is called an **analytical solution**, which is what we will spend most of our time on in this course.

For a given initial condition and a given force, the above procedure guarantees a unique solution. This fact leads to the following observation.

What it means to solve Newton's equation

Suppose we are given a Newton's equation with the total degrees of freedom $= M$. If we somehow manage to find an analytical solution which include $2M$ integration constant symbols, then that is *the* general solution! Conversely, the general solution for such an equation should have exactly $2M$ integration constant symbols. It does not matter how we find the solution (even if it was a guesswork ... mmm ... I mean an *educated* guesswork!). The $2M$ integration constant symbols would be replaced with numerical values, if the initial condition or equivalent physical condition were specified.

One comment. As we saw before, $\vec{a} = \ddot{\vec{x}}$ is a linear operator $(d/dt)^2$ acting on the function $\vec{x}(t)$. In other words, $\vec{a}$ is linear in $\vec{x}$. So, in $\vec{F} = m\vec{a}$, the right hand side is linear in $\vec{x}$. If the force is also linear in $\vec{x}$, then we say that we have a linear equation of motion. If not, then we have a non-linear equation of motion.