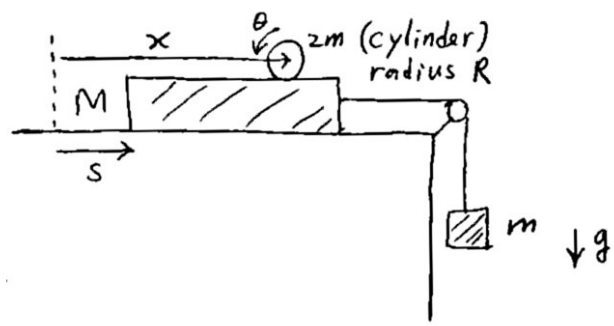
Due Nov. 16, Tuesday.

Problem 1 (30 points) Consider problem 5 of the pre-midterm. It involves three masses: m (mass on string, pulled by gravity), M (block on a frictionless horizontal surface, connected to mass m by a massless string through a massless and frictionless pulley), and $2m$ (cylinder with radius R, on M).



Initially all masses are at rest, while releasing m triggers the acceleration of all masses. The mass $2m$ rolls without slipping on M. [We consider only the first part of motion in which the mass $2m$ stays on top of mass M and the mass M remains on the horizontal surface. Eventually, of course, M will fall down, and $2m$ will drop off of M, but we are not concerned with such a long term fate of this system.] Two ways to do this problem has been shown in **an announcement at ecommons (see the file Sams-Solution.pdf)** (the other two files are for discussion only and should not be used for the basis of your work here). Here, we shall use the Lagrangian formulation given in that solution, but introduce a Lagrange multiplier for the constraint.

The horizontal coordinate for mass M can be defined as s, measured from its rest position. Likewise, the horizontal coordinate of the center of mass of $2m$ can be defined as x, measured from a fixed point on the table (not on the block M). Thus, both s and x are measured in the reference frame of the table (“lab reference frame”). The angular rotation of the cylinder around its axis can be measured by the angular coordinate θ. Using the convention that $\theta > 0$ for counter-clock-wise rotation, and $x, s > 0$ for the displacement to the right, the rolling without slipping constraint is given as

$$\dot{s} - \dot{x} \; = \; R\dot{\theta}$$

(a) Explain why the above constraint means the “rolling without constraint.” Namely, show that the velocity of the cylinder relative to the block M is zero at the contact point.

(b) Find the Lagrangian, $L(s, x, \theta, \dot{s}, \dot{x}, \dot{\theta}, t)$. Do not use the above constraint to reduce the number of generalized coordinates from three ($q_1 = s, q_2 = x, q_3 = \theta$) to two, as we are about to introduce the Lagrange multiplier for the above constraint. $I = mR^2$ for the cylinder (since mass $= 2m$).

(c) The above constraint is of the kind that we considered in class.

$$a_t + \sum_i a_i \dot{q}_i \; = \; 0$$

Find coefficients a_t and a_i's. [Your answer is not unique. It is arbitrary up to an overall multiplicative constant that can be applied to all coefficients.] You may use, for the subscript i of a_i, s, x, θ, instead of $i = 1, 2, 3$.

(d) Find the Lagrange equation of motion with constraint, for each $i = 1, 2, 3$ (or s, x, θ). By doing so, also find the generalized force of constraint $Q_i = a_i \lambda(t)$ (cf. the hint in part b of the next problem).

(e) Discuss the physical meaning of $\lambda(t)$ and each constraint force, Q_i, found in (d). Also, note that all Q_i's inter-dependent. Explain the details of their inter-dependence using Newton's 3rd law and Newton's 2nd law ($\vec{F} = m\vec{a}$ for the center of mass, and the net torque around the center of mass $= I\ddot{\theta}$). For each force or torque, be specific as to which object exerts that force/torque on which object.

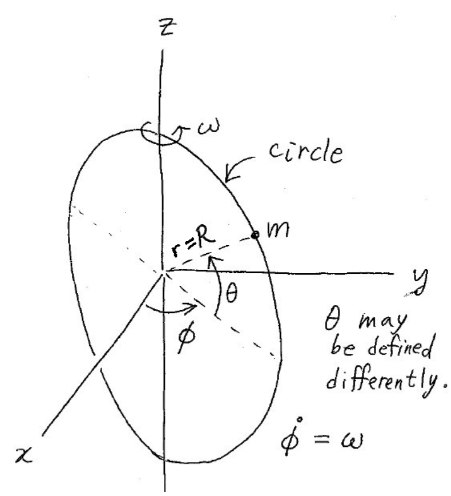
(f) Solve the three equations of motion found in (d), plus the equation of constraint, for $\ddot{s}$, $\ddot{x}$ and $\lambda(t)$. [Hint: You need to use the time-derivative of the equation of constraint, rather than the equation of constraint itself. There are four unknowns, $\ddot{x}, \ddot{s}, \ddot{\theta}, \lambda(t)$, and four equations, so the problem is solvable. The procedure is equivalent to the Newtonian method in the solution already provided in ecommons.]

(g) Discuss and compare the roles of two generalized forces of constraint (force Q_x and torque Q_θ) found in this problem with those in the example of a cylinder rolling down an incline (Lecture note 11, page 4). Namely, does the constraint force make the cylinder go faster or slower? Does the constraint torque make the cylinder go faster or slower?

(h) Consider the following statement. **For a rolling without slipping, the contact point is stationary, and thus despite the fact that a friction (a non-conservative force) acts on that point, the friction does no work, and thus the total mechanical energy of the rolling object is conserved**. This is an important piece of truth for a rolling without slipping (however, see note below). It is applicable, e.g., in the case of an object rolling down a hill (Lecture note 11, page 4). And yet, in the current problem, this statement just cannot be true, because the energy of the cylinder is clearly not conserved: it actually increases, since the kinetic energy increases while the potential energy is constant! Resolve this "paradox" by clarifying why the above statement is not applicable. Then, calculate the power delivered by the constraint force Q_x to the cylinder and show that it is equal to the time derivative of the total energy of the cylinder.

[Note: Here again, we are assuming that the rolling object has no deformation in shape, an idealized situation. Real objects do have deformation, and this contributes to some energy loss for *any* rolling. This was mentioned briefly in class.]

Problem 2 (30 points) A particle is constrained to move without friction on a circular wire, with radius R, rotating with constant angular speed ω about a vertical diameter. The whole setup is immersed in a constant downward gravitational field g. That is, the system under consideration is the same as that in problem 3 of Homework 5.



However, we now consider this problem as a Lagrangian mechanics problem with Lagrange multipliers, $\lambda_k(t)$, for constraints ($k = 1, 2$).

Constraint 1:

$$\dot{\phi} \;=\; \omega$$

Constraint 2: (because $r = R$)

$$\dot{r} \;=\; 0$$

In class we studied the theory with only one constraint/Lagrange-multiplier. Here, we have to deal with two constraints. This is no problem. It can be handled in two ways. (1) The formalism with one constraint is still applicable here, if one decides to solve the problem twice, each time with only one Lagrange multiplier ($\lambda(t)$) in the equation of motion, first for constraint 1 and then the second time for constraint 2. (2) However, we can also use the following, more general, Lagrange equations of motion with multiple constraints

$$-\frac{\partial L}{\partial q_i} + \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \;=\; \sum_{k=1}^{s} \lambda_k(t) a_{i,k}$$

where the constraints are ($k = 1, 2, \ldots, s$)

$$a_{t,k} + \sum_{i=1}^{n} a_{i,k}\dot{q}_i \;=\; 0$$

and $i = 1, 2, \ldots, n$ is the index for the generalized coordinates. Deriving the above Lagrange equations of motion with multiple Lagrange multipliers for multiple constraints is left for your exercise. (The derivation is very similar to that for the single constraint case, given during lecture and in lecture note.) For the rest of this problem, we will assume that the approach (2) is used.

For the generalized coordinates, we may define $q_1 = r$, $q_2 = \theta$, and $q_3 = \phi$. Then, we may use, when clarity requires it, the symbols r, θ, ϕ and the indices $i = 1, 2, 3$ interchangeably, as far as the subscripts are concerned. For instance, $a_{r,1}$ may be preferred to $a_{1,1}$.

(a) Find $a_{t,1}$, $a_{r,1}$, $a_{\theta,1}$, $a_{\phi,1}$ and $a_{t,2}$, $a_{r,2}$, $a_{\theta,2}$, $a_{\phi,2}$.

(b) Find the equations of motion. By doing so, also find the expression for the force of constraint Q_i

$$Q_i \;=\; \sum_{k=1}^{s} \lambda_k(t) a_{i,k}$$

for each $i = r, \theta, \phi$. [Hint: Be sure to simplify each equation of motion by using the constraints. For θ, the answer, which you should derive, is $Q_\theta = 0 = \frac{1}{2} m R^2 \sin 2\theta\, \omega^2 + mgR\cos\theta + mR^2\ddot{\theta}$.]

(c) Discuss the physical meaning of Q_r and *each* term in Q_r. Be specific as to which object exerts Q_r and what kind of "contact force" it is (normal force or friction). When discussing each term in Q_r, be mindful of distinguishing between the centripetal force and the centrifugal force.

(d) Does Q_r do any work on mass m? From your answer, find P_r, the power delivered by Q_r to the mass m.

(e) Discuss the physical meaning of Q_ϕ. Clearly specify whether it is exerted by the circular wire or some other external agent, left unspecified in this problem. Find the power P_ϕ delivered by Q_ϕ to the mass m.

(f) From problem 3 of Homework 5, find $d(E - H)/dt$. (Do not re-derive H in this problem, where we are treating constraint(s) with Lagrange multiplier(s).

The important results for H that we obtained in class, $dH/dt = \partial H/\partial t$ and one of the canonical equations, do *not* apply when the generalized coordinates used in L are not completely independent.) Show that your answer is equal to $\sum_i P_i$, where P_i is the power delivered by Q_i (see (b), (d), and (e)) to the mass m. Based on this finding, discuss *the physical reason* why E is not conserved and why H is conserved in that problem 3 of Homework 5.

Problem 3 (5 points) Assume that the Earth's orbit around the Sun is circular. Suddenly, the Sun's mass decreases by half. What is the new orbit of the Earth? Will the Earth escape the Sun? [Hint: Start by considering the relations between E, T, and U for a circular motion.]

Problem 4 (15 points) Two stars, with masses m_1 and m_2, are circling each other under the influence of the gravitational force. The period of the motion is τ and the radius of the circle is R. Assume that, suddenly, the two stars lose their speeds completely. Subsequently, the gravitional force causes them to fall towards each other and crash.

(a) What are the speeds of the two stars (in terms of m_1, m_2, R), when the distance between the two stars is halved?

(b) Show that it takes $\frac{\tau}{4\sqrt{2}}$ before the stars crash. Assume that the radii of the stars are much smaller than R, and thus negligible.

Problem 5 (10 points) A satellite is orbiting the Earth in a closed orbit. The maximum speed of the satellite is v_{max} and the minimum speed is v_{min}. Find the eccentricity ε in terms of these two quantities. Also, find ε in terms of r_{min} and r_{max} of the motion.

Problem 6 (10 points) In class (of Nov. 9), we used Newton's laws of gravity to explain Kepler's laws. Here, we do the opposite. Assume that we know Newton's three laws of motion and thus the Lagrangian formalism of mechanics also, but that we do not know Newton's law of gravity. Let us use Kepler's laws to figure out the nature of the gravitational force $\vec{F}(\vec{r})$, where $\vec{r} = \vec{r}_1 - \vec{r}_2$, $\vec{r}_1$ is the position of a planet, and $\vec{r}_2$ is the position of the Sun.

(a) Show that Kepler's first law and second law mean that the gravitational force between the Sun and a planet must be a central force: $\vec{F}(\vec{r}) = F(r)\hat{r}$. [Hint: Use the angular momentum conservation. From Kepler's first law, the fact the orbit belongs in a plane is all we need.]

(b) Show that Kepler's third law, if assumed to apply to a circular orbit with any radius, means that the central force must be a $1/r^2$ force: $F(r) = -k/r^2$.

[Note: If one uses Bertrand's theorem, then Kepler's first two laws and a physical argument (the magnitude of the force increases when the distance decreases) are all we need to prove, as we do here, that $\vec{F}(\vec{r}) = -k\hat{r}/r^2$.]