Due Nov. 2, Tuesday.

Problem 1 (10 points) **Fermat's principle and the PoLA.** Let us consider a photon, the quantum of light, with the angular frequency ω. While the study of photons is definitely not a realm of classical mechanics, some part of it can be understood using classical mechanics. The famous "Fermat's principle" of geometric optics is one of them. This principle states that "the light travels in a path that minimizes the time spent." This law, just like the principle of least action (PoLA), need to be re-stated to be valid in general cases. This is done in (a). Then, we study a simple case, in parts (b,c), where Ferma's principle in its old form can be derived from the PoLA.

(a) Consider a photon of angular frequency ω moving in a medium characterized by a position dependent refractive index $n(\vec{r})$. Write down the principle of least action, as we defined it in class, applied to this problem. Show that it means that the integral $\int_1^2 n(\vec{r})ds$ is stationary for the true path, where s measures the distance traveled.
Some potentially helpful remarks are offered now. At this point, you should not think about Fermat's principle at all. Just think PoLA. To figure out the Lagrangian, note that the kinetic energy of a photon is given as (from Phys 101)

$$T = pv = \hbar k v$$

where k is the wave number $(2\pi/\lambda)$, and the potential energy of a photon is zero. Here, $\hbar$ is the Planck constant $(h/(2\pi))$, v is the speed of light in the medium:

$$v = \frac{c}{n}$$

where c is the speed of light in vacuum. So, v is a position dependent function. This means k is position dependent as well, since

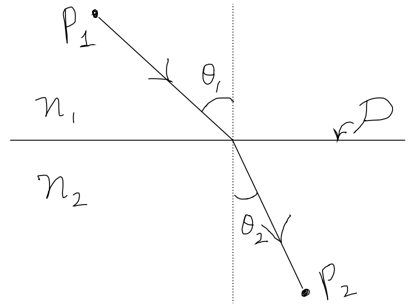
$$kv = \text{position independent}$$

is equal to its value in vacuum (due to the energy conservation).

(b) Consider a simple case when the space is divided into two parts by a boundary plane ($\mathcal{P}$). One side of plane $\mathcal{P}$ has a constant refractive index n_1 and the other side another constant refractive index n_2. Consider two arbitrary points P_1 and P_2, one in the n_1 region and the other in the n_2 region. It follows that the plane that contains P_1, P_2, and is perpendicular to $\mathcal{P}$ is uniquely determined. Within this plane, the following diagram can be drawn. Derive, by *minimizing* the action obtained in (a), Snell's law:

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Your solution should show your analysis of this problem with respect to all variational paths in three dimensions.



However, this is *not* to be a complicated calculus of variation problem. First of all, you can take it for granted (based on the intuition) that the shortest distance between two points in space (that is, the Euclidean space) is achieved by a straight line. Consider a point P_3, which is an arbitrary point in the plane $\mathcal{P}$ and which the path of the photon goes through. Then, all you need to do is to analyze the behavior of the action S as a function of P_3.

(c) Show that the function that is minimized in (b) is the total time of the motion, up to a multiplicative constant.

Problem 2 (10 points) **The *brachistochrone* problem and the generalized coordinate.** The so-called **cycloid** is a curve that is generated by a point of a rolling wheel. Assume that you are in a room with a ceiling whose vertical crossection is a cycloid shape:

$$\begin{aligned} x_c &= a(\phi + \sin\phi) \\ z_c &= a(1 - \cos\phi) \end{aligned}$$

Here ϕ is a real number, and z (x) is the vertical (horizontal) coordinate. Consider a pendulum consisting of a thin string of fixed length $4a$, attached to the ceiling at the position $a\pi\hat{x} + 2a\hat{z}$ (corresponding to $\phi = \pi$), and a mass m attached at the other end of the string. When this pendulum is allowed to oscillate in the xz plane, one can show, using a bit of calculus and trigonometry, that the resulting position (x, z) of the mass follows the path of a cycloid as well (you will be given an extra credit, if you actually prove this):

$$\begin{aligned} x &= a(\phi - \sin\phi) \\ z &= a(\cos\phi - 1) \end{aligned}$$

Now, ϕ should be thought of as a dynamical variable that describes the pendulum motion. The stable equilibrium position corresponds to $\phi = \pi$. For a given energy value, the maximum value of $|\phi - \pi|$ can be defined as the "amplitude" of the oscillation. The amplitude $\leq \pi$.

(a) Write down the Lagrangian for this pendulum motion.

(b) Re-express the Lagrangian in terms of the generalized coordinate $q \equiv \cos(\phi/2)$.

(c) Use the Lagrange equation of motion for q, to prove that the period of the oscillation *is independent of the amplitude* of the oscillation, and is $4\pi\sqrt{a/g}$.

[Note: The cycloid is also the solution of the *brachistochrone* problem, the problem of finding the path that takes the shortest time between two points in a constant gravitational field (cf. Example 6.2).]

Problem 3 (20 points) A particle is constrained to move without friction on a circular wire, with radius R, rotating with constant angular speed ω about a vertical diameter. The whole setup is immersed in a constant downward gravitational field g.

(a) Construct the Lagrangian, L, using the angular position θ around the circle as the generalized coordinate.

(b) Find the canonical momentum for θ. What is the physical meaning of it? Show that the Hamiltonian, H, is conserved, and is not equal to energy $= T + U$.

(c) However, one can define the effective kinetic energy, K, and the effective potential energy, V, so that $L = K - V$ and $H = K + V$. Identify K and V. Discuss each term in V.

(d) For θ from 0 to 2π, find all stable and unstable equilibrium points for V. Find the expansion of V around each of them up to the 2nd order. In the process, note that the shape of V is dependent on the value of ω. Identify the value ω_c that divides two different regimes $\omega < \omega_c$ and $\omega > \omega_c$.

(e) Obtain the angular frequency for small oscillation around each stable equilibrium point found in (d).

Problem 4 (15 points) Consider a spherical pendulum consisting of a bob of mass m attached to a weightless rod of length l, whose other end is attached to a pivot point. The pendulum pivots freely in all directions, and is in a constant gravitational field.

(a) Find the Lagrangian in the spherical coordinate system.

(b) Find canonical momenta p_θ, p_ϕ for θ, ϕ, respectively. For each of the these momenta, comment on its meaning and whether it is conserved or not.

(c) Find the Hamiltonian of the system. Using the energy conservation and the conserved quantity found in (b), find a formal solution of the problem in an integral form (analogous to page 8 of Lecture note 3). Identify an effective 1D potential, and show that it has exactly one minimum point when the conserved canonical momentum in (b) is non-zero. Discuss the nature of the motion at the minimum point.

Problem 5 (12 points) **Poisson bracket in Hamiltonian Mechanics.** We discussed the PoLA as an avenue to peer into Feynman's breathtaking view of Quantum Mechanics. Historically, though, this view of Feynman was not known for quite a while, as Feynman was not a first generation quantum physicist. At the time when Quantum Mechanics was just being born, the person who established the principles of Quantum Mechanics, by seeing through, and synthesizing, seemingly quite different equations discovered by Schrödinger and Heisenberg, was Dirac. Famously, Feynman recalls taking one of Dirac's footnotes seriously to kick-start his own inquiry leading to the path integral view of Quantum Mechanics. In any case, in Dirac's formalism of quantum mechanics (and this formalism is the standard Quantum Mechanics as we know it), the Poisson bracket plays the central role for the transition from Classical Mechanics to Quantum Mechanics. Here, we learn a bit about why this was the case. The Poisson bracket is defined in Hamiltonian Mechanics. It is

$$[g,h] \;=\; \sum_i \left(\frac{\partial g}{\partial q_i}\frac{\partial h}{\partial p_i} - \frac{\partial g}{\partial p_i}\frac{\partial h}{\partial q_i}\right)$$

where q_i is the generalized coordinate, i is the index for the degrees of freedom, p_i is the canonical momentum for q_i, and g, h are arbitrary functions of q_i, p_i and t. Verify the following properties, where H is the Hamiltonian.

(a) $[g,h] = -[h,g],\ [g,g] = 0$

(b) $[p_i,p_j] = 0,\ [q_i,q_j] = 0$

(c) $[q_i,p_j] = \delta_{ij}$ (Kronecker delta).

(d) $dg/dt \;=\; [g,H] + \partial g/\partial t$

(e) $\dot{q}_i \;=\; [q_i,H],\ \dot{p}_i \;=\; [p_i,H]$

(f) A function g that does not depend explicitly on t, is a constant of motion if and only if $[g,H] = 0$.

[Note: The Poisson bracket is to Classical Mechanics, what the "commutator" is to Quantum Mechanics. Their properties are very similar. In Quantum Mechanics, the definition of the commutator is $[A,B] = AB - BA$, where A, B are quantum mechanical "operators." H, q_i, p_i become operators in Quantum Mechanics. All of the above properties have direct analogues in QM commutators (the only difference being the multiplicative constant $\hbar\sqrt{-1}$ appearing here and there in QM). In particular, the QM version of (d) is the so-called "Heisenberg equation of motion," although it was Dirac who first discovered such elegant interpretation of the Heisenberg matrix mechanics. The QM version of (c) is the canonical quantization condition, responsible for the Heisenberg uncertainty principle.]