Due Oct. 26, Tuesday.

Problem 1 (10 points) [Reading this problem is as important/easy as doing this problem.] Here, we shall consider a subtle but important point about what we discussed in class as a key measure of chaos. As we discussed, this is the Lyapunov exponent, λ, defined as

$$\delta Z = \exp(\lambda t)\varepsilon$$

(continuous case; Newtonian mechanics problems; page 1 of Lecture note 7) and

$$d_n = \exp(\lambda n)\varepsilon$$

(discrete case; maps; page 3 of Lecture note 7). Here, $\varepsilon \to 0$ is a slight difference in the initial conditions of two systems that are identical otherwise, and d_n or δZ is the phase space distance of the two systems as $n \to \infty$ or $t \to \infty$, respectively. As usual, we consider only bound motions (paragraph 2 of page 2 of Lecture note 7). As in class, we will tacitly assume a one dimensional phase space only. [In general, the number of Lyapunov exponents is equal to the dimension of the phase space. It is sufficient for only one of them to be positive for the system to show chaos.]

(a) In the above definition, $n \to \infty$ or $t \to \infty$ is a subtle statement, and is to be interpreted with care. You will see why when you answer these questions. (i) Show that for any fixed $\varepsilon > 0$, very small but finite, if the limit $n \to \infty$ (or $t \to \infty$) is literally taken in the above equation, then you must conclude that $\lambda \to 0$ or less for any bound motion. (ii) Given this finding, how should be the statement "$n \to \infty$" (or $t \to \infty$) interpreted/defined so that the Lyapunov exponent would be defined in the most useful way? The two choices are: "large, but not too large, n so that d_n (or δZ) remains very small (negligible compared to the total range of motion)" and "$n \to \infty$, literally."
[Hint: Your conclusion should be consistent with the following fact. **Strictly speaking, the Lyapunov exponent describes only the initial tendency of the system *towards* the anomaly, but not the anomaly itself**. The anomaly in this sentence refers to the intuitive definition of chaos – the complete separation of two motions in a finite time no matter how nearly identical their initial conditions are.]

(b) Evaluate the Lyapunov exponent, λ, for the following so-called "tent map" by considering two nearly identical initial conditions, both at finite distance from the point $x = 1/2$:

$$\begin{aligned} x_{n+1} &= \mu x_n && \text{if } 0 < x_n < 1/2 \\ x_{n+1} &= \mu(1 - x_n) && \text{if } 1/2 < x_n < 1 \end{aligned}$$

where $0 < \mu < 2$. Show that λ is given by $\ln(\mu)$, and, thus, is greater than zero if $\mu > 1$, which is the chaotic regime of this map.
[Note 1: You *can* use Equation 4.52 of the textbook, if you like, but on one condition. You have to study its derivation given in page 176 of the textbook, and answer the following question. Where in the derivation was the nuanced meaning of $n \to \infty$ and d_n, as discussed in (a), used implicitly?]
[Note 2: The two values chosen to be at finite distance from 1/2 are "typical values." For some non-typical values, e.g. those with one value precisely at 1/2, the Lyapunov exponent may not even be well-defined! But, this does not pose a problem. **The Lyapunov exponent is an average over the exponents for all possible initial states**.]
[Note 3: For a "piece-wise linear" map like this map or the baker's map, the Lyapunov exponent can be obtained analytically, as you are demonstrating here.]

Problem 2 (10 points) Consider Example 5.3 of the textbook, which we went through in class. This time, though, consider a point P on the z axis, $P = (0, 0, z)$.

(a) Find the gravitational potential, Φ, at P.

(b) Expand Φ, on the z axis and near the origin, up to the leading order correction of z/a, assuming $|z/a| \ll 1$.

(c) Show that the origin is a stable equilibrium point of Φ, when it is considered as a function of z, as in (a) or (b).

(d) Combine the finding in (c), and the finding in class (or Example 5.3). Summarize the nature of the origin as the equilibrium point of Φ (stable or unstable), when a small displacement along one of the three principal axes (x, y, and z axes) is considered in turn. Overall, then, what is the origin for function Φ: is it a maximum point, a minimum point, or a saddle point? Briefly explain your answer. No calculation necessary in this part.

Problem 3 (10 points) Imagine drilling a straight hole through the center of the Earth from the south pole to the north pole. Consider the mass density of the Earth as uniform. Drop a particle into the hole. What kind of motion will the particle experience, assuming that there is no other force than the gravity of the Earth acting on the particle? Express the time taken for the particle to go from the south pole to the north pole, in terms of G, M_E, R_E, where M_E and R_E are the earth mass and the earth radius, respectively. How long is that time in minutes (two sig-figs)?

Problem 4 (20 points) Consider a cylinder with uniform density and infinite length. Let the mass per unit length along the cylinder be A, and the radius of the cylinder be R.

(a) Find the field $\vec{g}$ due to the cylinder at any position (including inside the cylinder). [Use the Gauss law.]

(b) Find the potential Φ due to the cylinder. In the cylindrical coordinate system (ρ, ϕ, z), sketch Φ as a function of ρ starting from 0 to a value larger than R.

(c) Now consider the situation where there are two of these cylinders separated by distance $2D$ with $D > R$. For the sake of computation, we may place one at $D\hat{x}$ and the other at $-D\hat{x}$. The two cylinders are parallel to each other and the z axis (as above). Write down the potential Φ due to these two cylinders at an arbitrary position $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$.

(d) Expand the potential for part (c), for position near the origin, outside both cylinders, so that x, y, z are very small (compared to D). Discuss the stability of a particle put at the origin, in view of the form of the potential function.

Problem 5 (10 points) Show that the gravitational "self energy" Σ_G for a uniform sphere of mass M and radius R is

$$\Sigma_G = -\frac{3}{5}\frac{GM^2}{R}$$

The "self energy" can be defined as the energy of assembly piecewise from infinity. The assembly process is thought to be a very very slowly so that the there is no kinetic energy involved. Or, it can be considered a virtual process, not a real process. A virtual process is a process that we imagine occuring with time held fixed.