Due Oct. 19, Tuesday.

Problem 1 (10 points) Near an unstable equilibrium point, x_0, of a one dimensional motion, the potential energy can be written as $U(x) = U_0 - \frac{1}{2}k(x - x_0)^2$, where k is a positive constant. We assume that there is no other force than the one given by $-dU(x)/dx$. Consider the motion with $E = U_0$. Find the general solution $x(t)$ for that motion in the close vicinity of x_0, assuming that the initial position is close to, but not exactly at, x_0. In particular, verify that your answer is consistent with the answer of Problem 4(g) of the previous homework. [Caution: In the solution of the 1D problem with $U(x)$, $t = \pm \int dx \ldots$, you should examine, separately, *both* solutions with differing signs for t, since the motion here is *not* periodic!]

Problem 2 (30 points) Consider a SHO (simple harmonic oscillator) driven by an external force. Assume that $f(t) = F(t)/m$ is an arbitrary function.

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t)$$

Here, $\beta \geq 0$, able to represent any physical cases (free, underdamped, critically damped and overdamped SHO). (As usual) we will be concerned with the particular solution only. In this problem, we will be looking for a particular solution that satisfies the following boundary condition

$$x(-\infty) = \dot{x}(-\infty) = 0$$

That is, the system is "quiet" at the far end of the past, before any $f(t)$ is "turned on."

(a) Let us write $x(t) = \exp(\alpha_1 t)g(t)$ where α_1 is one of the solutions for the characteristic equation for the above ODE (ordinary differential equation):

$$\alpha^2 + 2\beta\alpha + \omega_0^2 = 0.$$

Find the ODE satisfied by $g(t)$. It should be directly integrable over time. Do the integration from $-\infty$ to t, and use the above boundary condition, to show that $g(t)$ satisfies the following ODE, where α_2 is the other solution (which could be identical with α_1) of the above characteristic equation.

$$\dot{g} + (\alpha_1 - \alpha_2)g = \int_{-\infty}^{t} ds\, f(s)e^{-\alpha_1 s}$$

(b) Now, define $h(t)$ by
$$g(t) = h(t)e^{(\alpha_2 - \alpha_1)t}$$

and show that the ODE for $h(t)$ is directly integrable. Show that $h(t)$ is given by

$$h(t) = \int_{-\infty}^{t} ds \int_{s}^{t} dt' e^{(\alpha_1 - \alpha_2)t'} e^{-\alpha_1 s} f(s)$$

[Hint: You need to use $\int_{-\infty}^{t} dt' \int_{-\infty}^{t'} ds \cdots = \int_{-\infty}^{t} ds \int_{s}^{t} dt' \cdots$. Draw a diagram to show that this is true.]

(c) Show that $x(t)$ can be written as

$$x(t) = \int_{-\infty}^{t} dt' \, G(t - t') f(t')$$

where

$$G(t - t') = \theta(t - t') \int_{0}^{t-t'} du \, e^{(\alpha_1 - \alpha_2)u} \, e^{\alpha_2(t-t')}$$

and $\theta(t - t')$ is the Heaviside step function (1 if $t > t'$ and 0 if $t < t'$).

(d) (Extra credit) Show that if $f(t) = \delta(t - t_0)$ (the Dirac delta function – please look it up if you are not familiar with it), then

$$x(t) = G(t - t_0)$$

Such a function – the response of a system upon a unit delta function impulse – is generally called the **Green's function**. It is a concept that permeates throughout all physics. [Note: $f(t)$ should have the physical dimension of force/mass, different from 1/time. So, what does it mean that $f(t) = \delta(t - t_0)$? We assume that we have chosen the units so that the strength (i.e. the integral) of $f(t)$ is 1.]

(e) Show that for an under-damped or free SHO

$$G(t - t') = \frac{\theta(t - t')}{\omega_1} e^{-\beta(t-t')} \sin(\omega_1(t - t'))$$

where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$.

(f) Show that for a critically damped SHO

$$G(t - t') = \theta(t - t')(t - t') \exp(-\beta(t - t'))$$

(g) Show that for an over-damped SHO

$$G(t - t') = \frac{\theta(t - t')}{\gamma} e^{-\beta(t-t')} \sinh(\gamma(t - t'))$$

where $\gamma = \sqrt{\beta^2 - \omega_0^2}$.

This problem presented an important general method to obtain a particular so-lution for the driven SHO problem. By plugging in $f(t) = f_0 e^{i\omega t}$, $x(t)$ for a sinusoidal force can be re-derived. Any other function $f(t)$ can be readily han-dled.

Problem 3 (10 points) Using the solution of the previous problem, find a particular solution for $f(t) = f_0 \cos(\omega_0 t + \delta)\theta(t)$ for a SHO without any damping and natural frequency ω_0. [So, it corresponds to the resonance of a free SHO.] Here, θ means the step function, as in the previous problem. Your solution should consist of more than one term. Which term corresponds to the dominant term in the long run? Focus on that term. Is the energy conserved? If not, why not?

Problem 4 (20 points) Consider the van-der Pol oscillator equation (dimensionless form).

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

Do either one of the two parts below. If you do both, thank you and good job, but you will not get the double credit. This is because we are not requiring computer skills as prerequisite for this course.

(a) Show, using perturbation, that, when μ is very small, the solution con-verges to a limit cycle, which is a circle of radius 2 centered at the origin, in the (x, y) phase space, where $y \overset{def}{=} \dot{x}$. Do this by calculating dr/dt, perturbatively, where $r = \sqrt{x^2 + y^2}$. Here are some hints. First, note the equation of motion for x and y (see (b)). Second, express dr/dt as a func-tion of μ, x, y, given that equation of motion. Third, use the zero-th order solution, $x = A\cos t$ and $y = -A\sin t$ (counter-clock wise rotation, initial phase 0; this is general enough), in the expression that you got for dr/dt, to obtain the leading order solution for dr/dt. Show that over a long time scale (do the time average using the result of problem 2 of Homework 2) the effect of $dr/dt \approx dA/dt$ is to reduce A to 2.

(b) Show numerically, using your favorite computer environment (mathemat-ica, python, etc.), that the solution of this equation converges to a limit cycle regardless of the value of $\mu > 0$. (The motion for $\mu = 0$ is already periodic, of course.) Choose the initial condition so that it is not already on a limit cycle. Pick two (or slightly more?) values of μ, including μ much smaller than 1 and μ much greater than 1. In particular, when μ is very small, then the limit cycle should be a circle of radius 2 (see (a)). Numerically, you would need to integrate the following ODE.

$$\dot{x} = y$$
$$\dot{y} = -x - \mu(x^2 - 1)y$$

You do *not* need to write a routine to do this integration. Many routines, some free, are available for this job. Mathematica provides "NDSolve" to do this job for you. In python, "scipy.integrate.odeint" does the same. **For your solution on this part, if you choose to do this part, I require that you email your files (source code and results) to me. You should/need not submit any paper copy.**