Due Oct. 12, Tuesday.

Problem 1 (10 points) For each of the following statements, prove it if it is true, or provide one counter-example if it is not true. [*Note:* In this problem, we are considering functions of x and operators on those functions. If you like you can consider functions of t. I.e., you can change all x's to t's below, if you like. Changing x to t is more in line with Newton's equation (then $f(t)$ can be $x(t)$ or $v(t)$).]

(a) If L_1, L_2 are linear operators, then $\alpha L_1 + \beta L_2$ is also a linear operator, for any numbers α, β. [Note: $(L_1 + L_2)f(x) \stackrel{def}{=} L_1 f(x) + L_2 f(x)$ and $(\alpha L)f(x) \stackrel{def}{=} \alpha g(x)$ where $g(x) = Lf(x)$.]

(b) If L_1, L_2 are linear operators, then $L_1 L_2$ is also a linear operator. [Note: $L_1 L_2 f(x) \stackrel{def}{=} L_1 g(x)$ where $g(x) = L_2 f(x)$.]

(c) $Lf(x) = \exp(if(x))$. L is a linear operator.

(d) $Lf(x) = -\frac{d^3}{dx^3} f(x) + 8f(x)$. L is a linear operator.

(e) $Lf(x) = f(x)^2$. L is a linear operator.

(f) $Lf(x) = f(x) - 3$. L is a linear operator.

Problem 2 (5 points) Prove that

$$\frac{1}{\tau_n} \int_{t_0}^{t_0+\tau_n} dt \sin^2(\omega t - \delta) = \frac{1}{2}$$

$$\frac{1}{\tau_n} \int_{t_0}^{t_0+\tau_n} dt \cos^2(\omega t - \delta) = \frac{1}{2}$$

where $\tau_n = n\tau/2$ (n = integer), $\tau \stackrel{def}{=} 2\pi/\omega$ (period), and t_0, δ are any real constants.

Problem 3 (10 points) Consider the simple harmonic oscillator (SHO) problem $m\ddot{x} = -kx$ and its general solution $x = A\cos(\omega_0 t + \theta_0)$.

(a) Express the kinetic energy T as a function of t, m, ω_0, A, and θ_0.

(b) Express the potential energy U similarly.

(c) Add the two expressions and show, explicitly, that $E = T + U$ is is indeed a constant of motion.

(d) Define $\langle T \rangle$ as the time-average (over a multiple of the period) of the kinetic energy $\langle T \rangle$. Find the relationship between $\langle T \rangle$ and E.

(e) Define $\langle U \rangle$ similarly. Find the relationship between $\langle U \rangle$ and $\langle T \rangle$.

(f) Look up the "virial theorem" for power law forces, state it, and verify the answer you just obtained. According to the virial theorem, what would be the relationship between $\langle T \rangle$ and $\langle U \rangle$ if the equation of motion changed to $m\ddot{x} = -kx^3$?

Problem 4 (15 points) Consider a simple pendulum consisting of a massless rod of length l with a mass m attached at the end.

(a) Effectively there is only one degree of freedom for this motion, the angle θ. Write the potential energy and the kinetic energy in terms of $\theta, \dot{\theta}, m, l, g$.

(b) What values of θ correspond to the stable equilibrium? What values of θ correspond to the unstable equilibrium?

(c) Show that $-dU/d\theta$ is equal to the torque ($\vec{r} \times m\vec{g}$) on mass m.

(d) Write down the integral expression for the period τ of this motion for any angular amplitude $\Theta < \pi$. [Hint: lecture note 3, pages 8-9]

(e) Evaluate the integral expression in the limit of $\Theta \ll 1$ (in radians), and show that $\tau \approx 2\pi\sqrt{l/g}$ plus a small correction term. Evaluate the correction term up to the leading order. [Hint: $\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots$.]

(f) According to the result that you just obtained, did the correction term increase or decrease τ relative to the SHO value $2\pi\sqrt{l/g}$? Explain, using qualitative physics arguments of the simple pendulum, why your answer is correct.

(g) Show that $\tau \to \infty$ as $\Theta \to \pi$ from below.

Problem 5 (15 points) **A circular motion model of the driven SHO.** In the middle of a lake, a long pole stands. The pole is firmly built onto the bottom of the lake. It is strong and stable, and sticks just above the water. A bungee cord is connected to the top of the pole, and the other end of the bungee cord is attached to a jet ski. We do not consider any vertical motion of the jet ski. Only the horizontal motion is considered. The relevant forces acting on the jet ski are
(1) the elastic force from the bungee cord, which we assume to be

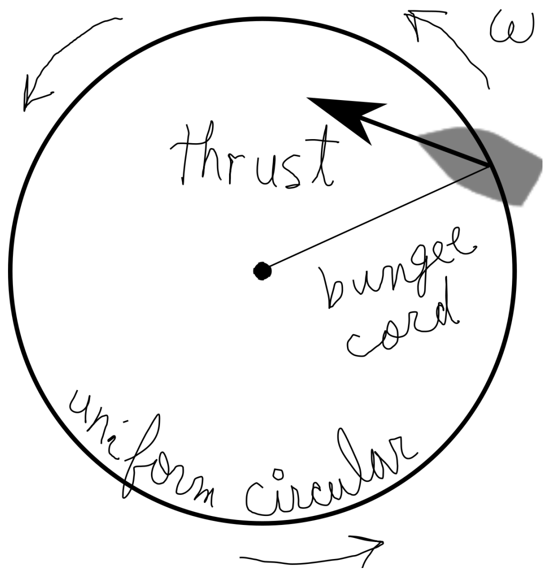
$$\vec{F}_b = -k\vec{r}$$

(2) the damping force from the water, which we assume to be

$$\vec{F}_r = -b\vec{v}$$

(3) and the thrust of the engine of the jet ski, which we assume to be

$$\vec{F}_t = F_0 \left(\hat{x}\cos(\omega t) + \hat{y}\sin(\omega t)\right).$$



(a) Write down the x component (or the y component) of the equation of motion, and show that it has exactly the same form as that for the driven SHO problem.

(b) Consider a steady state of the jet ski. Assume that the jet ski's motion is a circular motion, among all possible motions. The position vector of the jet ski is given, with a phase shift/lag δ, as follows.

$$\vec{r} = D(\hat{x}\cos(\omega t - \delta) + \hat{y}\sin(\omega t - \delta))$$

Draw a free body diagram for this motion. What is the geometrical meaning of δ in the free body diagram?

(c) Re-derive, by considering Newton's law associated with your free body diagram, the following key results of the driven SHO problem.

$$\delta = \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right) \qquad \beta = b/(2m)$$

$$D = \frac{F_0/m}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\beta^2\omega^2}}$$

(d) In this rotational motion model, explain, in physical terms (the centripetal force and the speed), why δ goes from 0 (for $\omega = 0$) to $\pi/2$ (for $\omega = \omega_0$) to π (for $\omega \to \infty$).

Problem 6 (10 points) A grandfather clock has a pendulum length (l) of 0.8 m, and mass bob (m) of 0.3 kg. A mass (M) of 2 kg falls 0.5 m (H) in a week to keep the angular amplitude of the pendulum oscillation steady at 0.03 rad. What is the value of Q of this system?