

Due Oct. 5, Tuesday.

Each problem is worth 5 points, except problems 5, 6, 7 (10 points each).

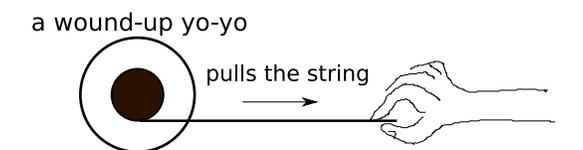
Problem 1 Assume that \vec{O}_1 and \vec{O}_2 are arbitrary orthogonal matrices of the same dimensions. For each of the following statements, prove it if it is true, or provide one counter-example (using 2×2 matrices is good enough) if it is not true.

- (a) $\vec{O}_1 + \vec{O}_2$ is also orthogonal.
- (b) $\vec{O}_1 \vec{O}_2$ is also orthogonal.
- (c) $r \vec{O}_1$ is also orthogonal, where r is an arbitrary real number.
- (d) \vec{O}_1^t is also orthogonal.

Problem 2 Given an arbitrary orthogonal coordinate transformation \vec{O} : $\vec{r}' = \vec{O} \vec{r}$ prove following statements. You can use the fact that the scalar product is a scalar, without proof, if necessary. You can consider the 3D space, specifically, but these statements are valid for any dimensions.

- (a) $\vec{v} = \dot{\vec{r}}$ is a vector. [Trivial, but show it. Stated in class, without really showing.]
- (b) The gradient operator $\vec{\nabla} \stackrel{def}{=} \sum_{i=1}^3 \hat{i} \frac{\partial}{\partial x_i}$ is a vector, where x_i is the i -th Cartesian component of the position vector \vec{r} ($x_1 = x$, $x_2 = y$, $x_3 = z$, in 3D) and \hat{i} is the i -th unit vector in the Cartesian coordinate system ($\hat{1} = \hat{x}$, $\hat{2} = \hat{y}$, $\hat{3} = \hat{z}$, in 3D). (cf. Section 1.6 of the text.)
- (c) The distance between two positions \vec{r}_1 and \vec{r}_2 is a scalar.
- (d) The magnitude of the relative velocity $\vec{v}_1 - \vec{v}_2$ is a scalar.

Problem 3 Reference frame. For rolling without slipping, it is often very convenient to consider a reference frame whose origin is taken at the contact point of the rolling at a particular instant. There is a fully wound yo-yo lying on a desk, as shown. You pull the string gently towards you, as indicated in the figure. The yo-yo rolls without slipping. Figure out which way the yo-yo starts to roll, based on Newton's laws (\vec{L} form and/or \vec{p} form).



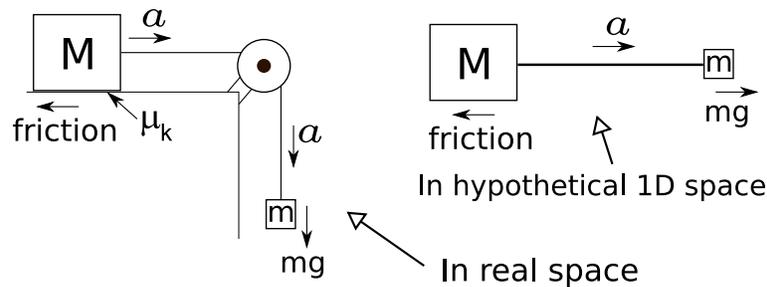
- (a) Do this problem using a reference frame whose origin is at the initial contact point between the yo-yo and the desk.

- (b) Do this problem using a reference frame whose origin is somewhere else, e.g. the initial position of the center of the yo-yo. [This part is more difficult, and may be worth extra credit.]

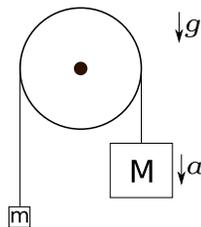
You need to list and discuss *all* forces acting on the yo-yo.

Problem 4 Straightening string trick. In problems involving a massless string on frictionless pulleys, all that the string does is transferring forces. In addition, if all masses are moving only parallel to the string, then the problem is effectively one dimensional, and a trick presented here can be very useful for calculating the acceleration (a) *extremely* easily. This is the trick. First, we re-map the geometry by straightening the string and, accordingly, rotating forces. Pulleys disappear. Second, we define all masses plus the string to be “a particle.”

- (a) Consider the two diagrams below, which illustrate this trick. The diagram on the right does not show a situation in real physical space. It merely shows an, artificial but useful, conceptual re-mapping of the diagram on the left. On the right diagram, take the two masses plus the string as one “body” or one “particle.” By calculating the net force on this body, express a in terms of μ_k , g , m and M .



- (b) Consider the Atwood machine (Example 2.9). Sketch an equivalent diagram in hypothetical 1D space, as in (a), and, again, obtain a in terms of g , m and M , without ever involving the tension force.



- (c) For each of (a) and (b), (i) show, in details, that your answer is correct dimension-wise (the dimension of a physical quantity, not the spatial dimension), (ii) show that your answer is correct when $M \rightarrow 0$, and (iii) discuss whether your answer is applicable when $M \rightarrow \infty$.

Problem 5 Consider tossing a ball upwards with initial speed v_0 . There is air resistance $-mkv^2$ as well as gravity (the negative sign here means that the air resistance is opposite in direction to the velocity). While this problem is completely solvable by brute force integration, here we won't do that. Instead, we will study this problem using the perturbation method only. The perturbation method is very useful for gaining insight, since it can be easily applied to any problem, whether it is exactly solvable or not.

- (a) What is the dimension of k ?
- (b) What is the terminal speed v_t ? [Determine it using a free body diagram.]
- (c) For small k , treat the air resistance as a small term in Newton's equation of motion (EOM), and calculate the leading order correction due to the air resistance for (i) the time that it takes for the ball to reach the maximum height, (ii) the maximum height, (iii) the time it takes for the ball to come back to the original vertical position, (iv) the speed when it comes back to the original vertical position. [Again, DO NOT solve the EOM exactly and then do a series expansion. Instead, use the perturbation method on the EOM. First, solve the EOM without the air resistance. Then, include the air resistance term, but plug in the first solution to the air resistance term of the EOM, to get the leading order correction to x .]
- (d) Use the work-energy theorem to re-derive, and thus verify, your answer for part (iv) of (c).
- (e) Make a sketch of your answer for part (iv) of (c), along with the exact answer (read prob 2-12 of the textbook). Compare the two curves and discuss.

Problem 6 A particle is released from rest and falls under the influence of gravity and air resistance. Find the relationship between v and the distance of falling y when the air resistance is equal to (a) αv and (b) βv^2 . In both cases, you should show that your solutions reduce to the expected formula when the air resistance is turned off. The trick $a = dv/dt = (dv/dx)(dx/dt) = v(dv/dx)$ may be useful for this problem.

Problem 7 Consider the projectile motion problem with air resistance $-mk\vec{v}$, as we solved it in class. In the limits of $\alpha \stackrel{\text{def}}{=} kv_{y,0}/g \ll 1$ and $\alpha \gg 1$, obtain the leading correction to the optimum throw angle (45 degrees if no resistance) due to the air resistance.

Problem 8 2-47 of the textbook.

Problem 9 2-53 of the textbook.