One letter-size crib sheet is allowed, with only one pageful of writing. No calculator. No other materials. Any questions to proctor are allowed.

$$
\begin{aligned}
\sin(a \pm b) &= \sin a \cos b \pm \cos a \sin b \\
\cos(a \pm b) &= \cos a \cos b \mp \sin a \sin b \\
\sin(2a) &= 2 \sin a \cos a \\
\cos(2a) &= 2\cos^2 a - 1 \;=\; 1 - 2\sin^2 a \\
\sin\delta &= \delta + O(\delta^3) \\
\cos\delta &= 1 - \frac{1}{2}\delta^2 + O(\delta^4) \\
(1+\delta)^\alpha &= 1 + \alpha\delta + \frac{1}{2}\alpha(\alpha-1)\delta^2 + O(\delta^3) \\
\ln(1+\delta) &= 1 + \delta - \frac{1}{2}\delta^2 + O(\delta^3) \\
e^\delta &= 1 + \delta + \frac{1}{2}\delta^2 + O(\delta^3)
\end{aligned}
$$

The last problem is an extra-credit problem, which will be counted towards supplementing your midterm score: it will be worth one problem of the midterm. The last part of each of problems 1,2,3 may be more time consuming than other parts. Plan your time accordingly.

The weight of the regular part of this exam (the first three problems) is adjustable. The weight of the regular part of this exam to the total score that will determine your letter grade is 30 % according to the syllabus. However, you can request a higher percentage, up to 50 %, if you believe that this exam better reflects your learning. You can also request a slightly lower weight, down to 25 %, if you like to minimize a potential negative impact of this exam. If you do make a request for an adjusted weight, then I will calculate your grade based on the two weights, the default value and the requested value, and choose whatever gives you a better grade. If your requested weight value is used, then all other scores (midterm, homework, quiz) of yours will be adjusted down or up uniformly. **You can make the request by including a sentence to that effect, at the beginning of your work.** You cannot make/modify the request after the exam is over.

Show all your work. **Take time to read this exam.** Be neat in writing.
Believe that you are the one!! Good luck!

Problem 1 Two bodies interact with a central force. We consider only the relative motion of the two bodies. The Lagrangian is given by

$$L = \frac{1}{2}\mu\dot{\vec{r}}\cdot\dot{\vec{r}} - U(r)$$

$\vec{r} = \vec{r}_1 - \vec{r}_2$, $r = |\vec{r}|$, and $\mu = (\frac{1}{m_1} + \frac{1}{m_2})^{-1}$ (the reduced mass).

(a) Show that the vector $\vec{r}$ is confined to a plane.

(b) Show that the area swept by $\vec{r}$ increases at a constant rate ("constant areal velocity") [thus proving the generalized version of Kepler's law for any central force and any trajectory]. The formula $dA = |\vec{r} \times d\vec{r}|/2$ may be used without proof, where dA is the change of the area during dt.

(c) Find $U_{eff}(r)$, in terms of which the energy can be written as

$$E = \frac{1}{2}\mu\dot{r}^2 + U_{eff}(r)$$

Note that $U_{eff}(r)$ should be a function of r only (no $\dot{r}$, θ or $\dot{\theta}$ in it!).

(d) Now consider a special case: the Kepler problem.

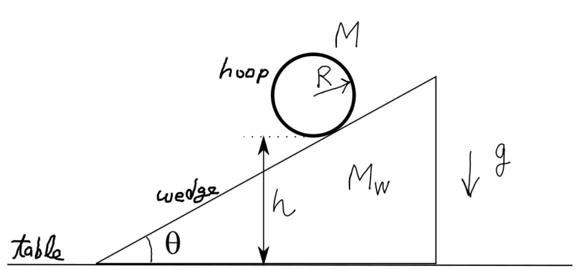
$$U(r) = -k/r \qquad (k > 0)$$

Suppose that you remembered correctly that the general shape of the orbit for this potential is given by

$$\alpha u = 1 + \varepsilon\cos\theta \qquad (\varepsilon \geq 0)$$

where $u \stackrel{def}{=} 1/r$. Find α and ε in terms of μ, k and the constants of motion. [Hints: Evaluating u and $u' \stackrel{def}{=} du/d\theta$ at two convenient values of θ (0 and $\pi/2$, for example) would be a good start. I am *not* asking you to integrate the equation of motion to derive $\alpha u = 1 + \varepsilon\cos\theta$!]

Problem 2 A hollow cylinder ("hoop") with radius R and total mass M is placed on a wedge of mass $M_W = 3M$ and incline angle θ, as shown in the figure. The constant gravitational field g points downward. The wedge is *free to slide on the table without friction.*



Initially, the cylinder is at height h, measured from the table to the lowest point of the cylinder, and zero total kinetic energy. The wedge also has zero kinetic energy, initially. The cylinder is then released and starts to roll down. The wedge slides. We consider the motion between $t = 0$ and $t = t_t$ only, where $t = 0$ is the time of release, and t_t is the time at which the cylinder touches the table. Assume that the cylinder rolls without slipping.

(a) Find the acceleration of the cylinder relative to the wedge, and relative to the table.

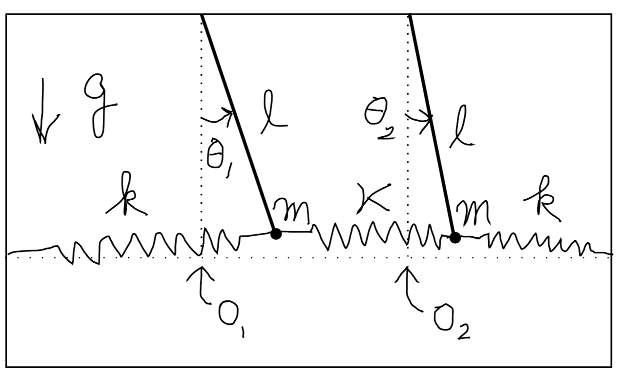
(b) Find t_t. Find the velocity of the center of mass of the cylinder and the velocity of the wedge, both at $t = t_t$ and both relative to the table. This part may be done after doing part (c), if you like.

(c) Find, and discuss the nature of, all constants of motion of this problem.

(d) Find the force of constraint that acts along the surface of the wedge that the cylinder rolls on. Be specific as to which object exerts this force on which object. Find the total work done by this force of constraint from $t = 0$ to $t = t_t$.

(e) Consider a solid cylinder, with the same mass M, but with radius r, which may not be equal to R. This solid cylinder is given the same initial condition and travels down the same height h before touching the table. Would t_t(solid) for this case be greater than or less than t_t(hollow) of the above case? Explain your answer, as precisely as possible, but avoiding any unnecessary calculation. Do *not* assume that the formula for the rotational inertia for the solid cylinder is known. If you think it is necessary to know it, then you should derive it.

Problem 3 Consider a coupled oscillator system consisting of two simple pendulums and three springs, connected as shown. The left (right) end of the first (third) spring is fixed to the left (right) wall. These two springs have the spring constant k, while the spring in the middle has the spring constant κ. In equilibrium, the two masses are at O_1 and O_2 respectively. The constant gravitational field g points downward.



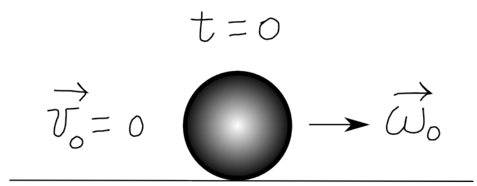
(a) Assuming small oscillations, find the Lagrangian L.

(b) Find the matrices $\overleftrightarrow{M}$ and $\overleftrightarrow{A}$, where $T = \frac{1}{2}\dot{\vec{q}}^t \overleftrightarrow{M} \dot{\vec{q}}$, $U = \frac{1}{2}\vec{q}^t \overleftrightarrow{A} \vec{q}$, $\vec{q}^t \stackrel{def}{=} (\theta_1\ \theta_2)$.

(c) Find the normal modes (the natural frequency and the relation between θ_1 and θ_2, for each mode).

(d) At $t = 0$, $\theta_1 = \alpha,\ \theta_2 = 0,\ \dot{\theta}_1 = 0,\ \dot{\theta}_2 = \beta$. Find $\theta_1(t)$ and $\theta_2(t)$ at all times.

Problem 4 (Extra credit) A billiard ball lies on a horizontal plane. A billiard wizard hits the ball from above (do *not* try it, if you are not such a wizard already – it is way too easy to tear apart the table cloth by mistake when trying such a shot!). At $t = 0$, the moment right after the hit, the ball is found to have an angular velocity parallel the plane but zero velocity of the center of mass (see the diagram). There is a finite coefficient of kinetic friction, μ, between the ball and the plane. On the other hand, we assume that, *if* and *when* the ball rolls without slipping, it experiences no friction. The constant gravitational field g points downward. Consider the horizontal plane to be infinite in extent.



(a) The ball has uniform mass density, radius R, and total mass M. Find its inertia tensor in the center of mass frame.

(b) Find and plot the velocity of the center of mass of the ball as a function of time ($t \geq 0$). Read the comment below.

(c) Find and plot the angular velocity of the ball around the center of mass as a function of time ($t \geq 0$).

(d) Find and plot T (total kinetic energy), T_M (kinetic energy associated with M at the center of mass), $T_{int,cm}$ (kinetic energy for the rotation around the center of mass) as a function of time ($t \geq 0$).

[**Comment for parts b,c,d**] There are two different regimes of motion as a function of time. Find the value of t_c, which separates the two regimes. Your plot should clearly mark t_c, and clearly show the values of the physical quantity plotted, at $t = 0$ and $t = t_c$.