Closed everything. No calculator. If you need other (physics or math) formula, you may ask if it can be given. Please ask if any problem needs to be clarified further.

$$
\begin{aligned}
\sin(a \pm b) &= \sin a \cos b \pm \cos a \sin b \\
\cos(a \pm b) &= \cos a \cos b \mp \sin a \sin b \\
\sin(2a) &= 2 \sin a \cos a \\
\cos(2a) &= 2 \cos^2 a - 1 \ = \ 1 - 2 \sin^2 a \\
\sin \delta &= \delta + O(\delta^3) \\
\cos \delta &= 1 - \frac{1}{2}\delta^2 + O(\delta^4) \\
(1+\delta)^\alpha &= 1 + \alpha\delta + \frac{1}{2}\alpha(\alpha - 1)\delta^2 + O(\delta^3) \\
\ln(1+\delta) &= \cancel{1} + \delta - \frac{1}{2}\delta^2 + O(\delta^3) \\
e^\delta &= 1 + \delta + \frac{1}{2}\delta^2 + O(\delta^3)
\end{aligned}
$$

Show all your work. Be as neat as possible in writing your solutions. Even if your answer for a later part of a given problem is incorrect, it can happen that this is mostly or totally due to the fact that your answer in the early part of that problem was off to a degree. In such a case, the logical steps from the early part to the later part may be given much, possibly full, credit. Likewise, a correct answer resulting from incorrect steps will be given only partial, in some cases very little, credit.

Do only one of problems 1-3 (40 points). Do only two of problems 4-6 (50 points each). Problem 7 is mandatory (60 points).

Problem 1 Explain why the phase lag δ of x, relative to a sinusoidal driving force F, is $\pi/2$ on resonance ($\omega = \omega_0$) for a damped simple harmonic oscillator with the natural frequency ω_0. Suggested method: either (1) the maximum power transfer argument or (2) the uniform circular motion picture ("jet ski on a bungee cord" picture) of the resonance. The third (brute force) method is deriving the full function $\delta(\omega)$, and evaluating it at $\omega = \omega_0$.

$$m\ddot{x} = -kx - b\dot{x} + F(t)$$

Problem 2 State the work-energy theorem and discuss for what forces it is applicable. Derive it from Newton's law, $\vec{F} = m\vec{a} = m d\vec{v}/dt$, and the definition of work.

Problem 3 Name the three fundamental kinds of symmetry of space and time. Explain what they mean for physical laws, what they mean for the Lagrangian of a closed system, and what conserved quantities they lead to.

Problem 4 A particle is dropped from rest.

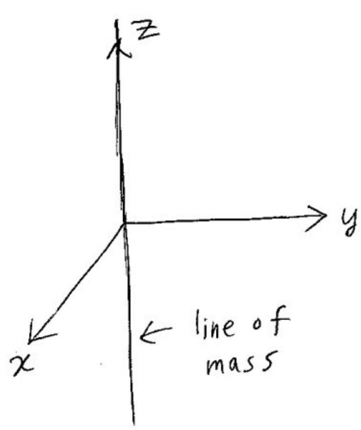
(a) Without air resistance, how long does it take for the particle to drop distance h? Your answer should be a function of h and g, the surface gravitational field.

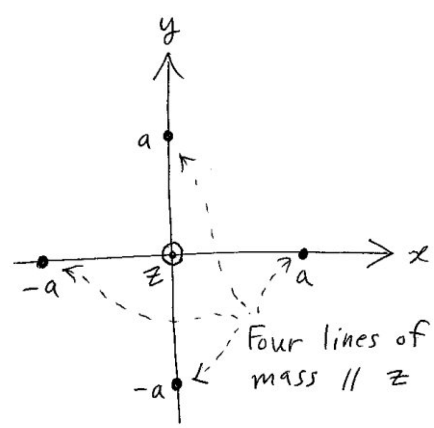
(b) With air resistance $-mkv$, how long does it take for the particle to drop distance h? Find your answer only up to the leading order correction of the air resistance, assuming that the air resistance is weak. [Hint: You can solve Newton's equation either (1) using the perturbation theory, or (2) exactly and then doing an expansion. Then, to solve for the time corresponding to the distance h, you would need to use the perturbation method.]

(c) What is the perturbation parameter in part (b)? Express your answer in terms of the answer for part (a) and k. [Hint: It must be dimensionless. Even if you don't have a correct answer for (b), a dimensionless parameter obtained by "an intelligent guess" is acceptable.]

Problem 5 This problem consists of three parts. If you are not able to do part (a), then you may still do parts (b,c) assuming point masses instead of line masses.

(a) There is a line of mass along the z axis, passing through the origin (the left figure below). The total mass M and the total length of the line L are infinite, but the mass per unit length is finite, $A = M/L$. Find the gravitational potential Φ, in terms of A and position coordinates. [Hint: Gauss law, $\int d\vec{S} \cdot \vec{g} \;=\; -4\pi G M$.]

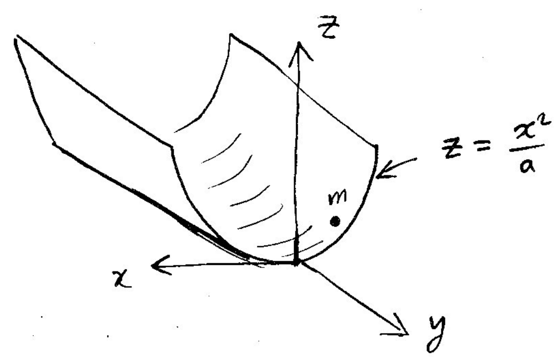




(b) There are four lines of mass, each identical with the one in (a) but placed at different positions. The top view is depicted in the right figure above (no line of mass at the origin – the circle and the dot at the origin are just for the z axis.) The lines of mass are parallel to the z axis, and are located at $(x, y) = (\pm a, 0)$ and $(0, \pm a)$. Find the gravitational potential Φ due to them.

(c) Expand the potential of (b), to 2nd order in x and y, assuming $|x|, |y| \ll a$.

Problem 6 A particle with mass m moves, without friction, on a "trough" whose shape is given by $z = x^2/a$. There is no friction. There is a constant gravitational field, $\vec{g} = -g\hat{z}$. The following diagram shows a bottom section of the trough. Note that the trough is infinite in length (in y) and height (in z).



(a) Find the Lagrangian.

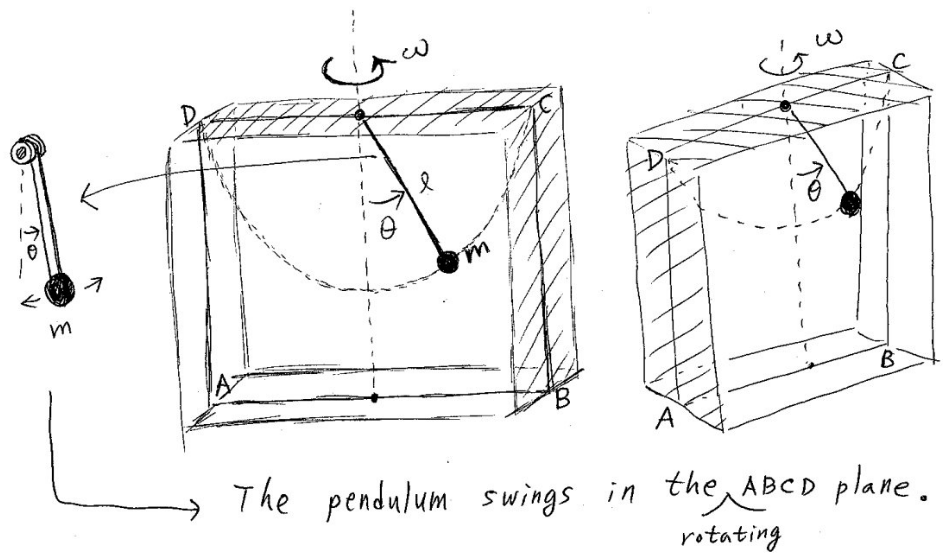
(b) Find the equations of motion for x and y.

(c) For small x, find the period of the x motion in terms of g, a. [Hint: $\dot{x}$ or $\ddot{x}$ is $O(x)$. Ignore terms that are higher order than linear in x in the equation of motion. For example, $x^2\ddot{x}$ would be such a term ($O(x^3)$) to ignore.]

(d) What are the conserved quantities among H, p_x, p_y, L_z? (H: Hamiltonian, p_x, p_y: linear momentum along x and y respectively, L_z: the z-component of the angular momentum.) Concisely explain why.

Flip the page for problem 7.

Problem 7 A simple plane pendulum is attached to the center of the ceiling of a room. The pendulum is constrained to oscillate in the plane ABCD. The pendulum swings without any damping. The room is rotating around the vertical axis at a constant angular velocity ω, as does the plane ABCD in which the pendulum oscillates. Note that $|\theta| < \pi/2$.



(a) Find the Lagrangian.

(b) Find the canonical momentum p_θ and the Hamiltonian H.

(c) Which of the quantities H, E, p_θ is/are conserved? For each quantity, explain briefly why it is conserved (or not). Here, $E = T + U$, where $T = \frac{1}{2}mv^2 = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2)$.

(d) In the reference frame of the room, what is the effective potential, $V(\theta)$, for this pendulum, given the above Lagrangian and Hamiltonian?

(e) Find all possible equilibrium points for $V(\theta)$. Show that some equilibrium points exist only when $\omega \geq \omega_c$. Find ω_c.

(f) Sketch $V(\theta)$, in the two regimes, $\omega < \omega_c$ and $\omega > \omega_c$. [Hint: If you are unsure how to proceed, then it may help to first examine the limits $\omega \to 0$ and $\omega \gg \omega_c$. Using the physical intuition, rather than the mathematical rigor, is allowed for producing the sketch.]