L + Constraint(s)

A constraint $\quad a_t + \sum_i a_i \dot{q}_i = 0 \quad \cdots (*)$

EoM $\quad -\frac{\partial L}{\partial q_i} + \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = a_i \lambda(t)$

(**) When using a $\lambda(t)$, a Lagrange undetermined multiplier, all q_i's are considered as free variables before setting up EoM's.

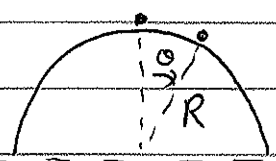
(***) After setting up EoM's, the constraint(s) can be used with EoM's to solve the problem. However, constraints should not be used before setting up EoM's when dealing with them through $\lambda(t)$'s!!

* If there are multiple constraints then

$$a_i \lambda(t) \rightarrow \sum_k a_{i,k} \lambda_k(t)$$

k ... constraint index

Ex.) Bug-on-head problem



Constraint : $r = R$

$\Rightarrow \dot{r} = 0$ ←--- in the form of (*)!

$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta$

$\Rightarrow a_t = 0, \; a_r = 1, \; a_\theta = 0$

comparing with (*)

EoM's : $-mr\dot{\theta}^2 + mgc_\theta + m\ddot{r} = \lambda(t)$

$-mgrs_\theta + \frac{d}{dt}(mr^2\dot{\theta}) = 0$

r not R!! Because of (**)!!

Now, having obtained EoMs, we are free to use the constraint $r = R$, to set $\ddot{r} = 0$

$\frac{d}{dt}(mr^2\dot{\theta}) = mR^2\ddot{\theta}$. (But not before setting up EoMs!)

$\therefore \lambda(t) = mgc_\theta - mR\dot{\theta}^2$