PH102, 2013W, Lecture Notes: January 23, Thurs, Class 6

Angular Momentum in External Magnetic Fields

Objectives:

- Differentiate how an electron's total angular momentum is constructed in the presence of strong and weak magnetic fields.

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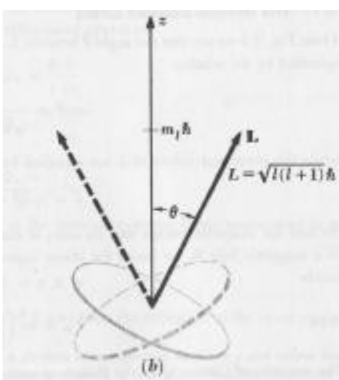
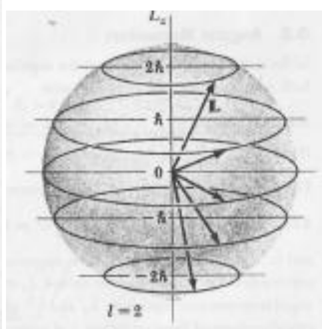
An electron revolving around a nucleus in an atom has two types of angular momentum:

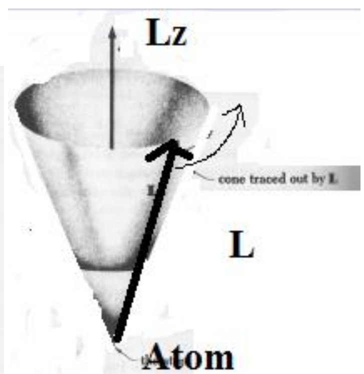
- Orbital angular momentum: $\vec{L}$
- Spin angular momentum: $\vec{S}$

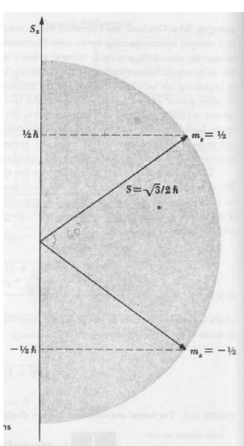
$$|\vec{L}| = \sqrt{l(l+1)}\hbar \qquad where\ l = 0, 1, 2, \dots n-1$$
$$L_z = m_l \hbar \qquad where\ m_l = -l, -l+1, \dots l-1, l$$

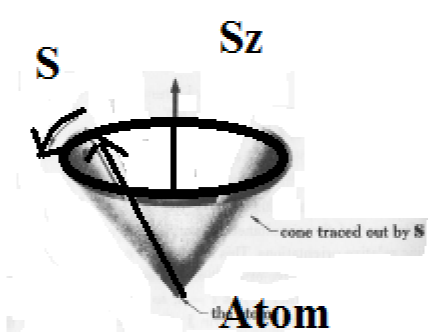
$$|\vec{S}| = \sqrt{s(s+1)}\hbar \qquad where\ s\ is\ a\ number\ intrinsic\ to\ a\ given\ particle$$
$$S_z = m_s \hbar \qquad where\ m_s = -s, -s+1, \dots s-1, s$$









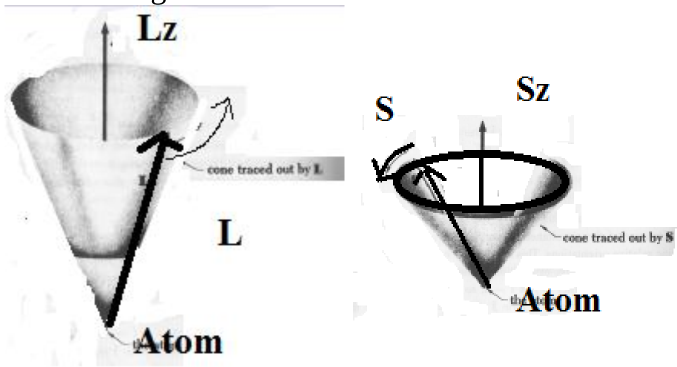


In the presence of an external magnetic field, $\vec{B}$, we need to consider

- Magnetic dipole moment related to orbital angular momentum of an electron, $\vec{\mu}_L = -\left(\frac{e}{2m}\right)\vec{L}$

- Magnetic dipole moment related to spin angular momentum of an electron, $\vec{\mu}_{\boldsymbol{S}} = -\left(\frac{e}{m}\right)\vec{S}$

The Paschen-Back Effect:
When the external magnetic field $\vec{B}$ is strong, then the $\vec{L}\ and\ \vec{S}$ are separately aligned with the external magnetic field.



Energy related to the magnetic moment due to $\vec{L}$

$$U_L = -\vec{\mu}_{\boldsymbol{L}.}\cdot\vec{B} = -\mu_{L_z}B_z = -\left(-\frac{e}{2m}L_z\right)B_z = \frac{e}{2m}m_l\hbar B_z$$

Energy related to the magnetic moment due to $\vec{S}$

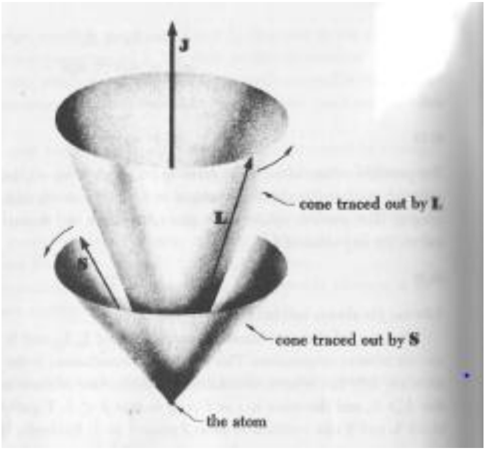
$$U_S = -\vec{\mu}_{\boldsymbol{S}.}\cdot\vec{B} = -\mu_{S_z}B_z = -\left(-\frac{e}{m}S_z\right)B_z = \frac{e}{m}m_s\hbar B_z$$

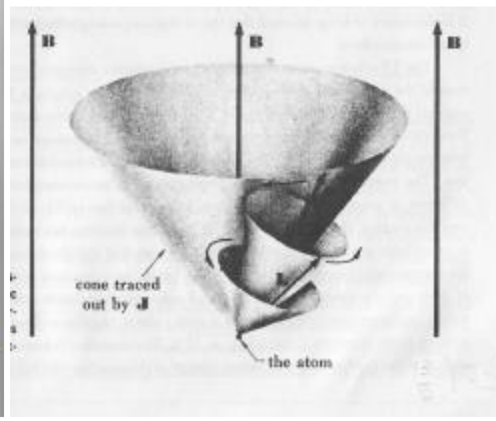
Therefore $E_n\ will\ split\ into$:

$$E_{strong\ magnetic\ field} = E_n + \frac{e}{2m}(m_l + 2m_s)\hbar B_z$$

In this case, n, l, m_l, m_s are good quantum numbers.

The Zeeman Effect:
When the external magnetic field $\vec{B}$ is weak, then the $\vec{L}\ and\ \vec{S}$ of an electron are coupled into a total angular momentum, $\vec{J} = \vec{L} + \vec{S}$





Therefore, the total magnetic moment can be written as

$$\vec{\mu}_{J.} = \vec{\mu}_{L.} + \vec{\mu}_{S.} = -\left(\frac{e}{2m}\right)(\vec{L} + 2\vec{S})$$

Note that $\vec{J} = \vec{L} + \vec{S}$ *and* $\vec{\mu}_{J.} = -\left(\frac{e}{2m}\right)(\vec{L} + 2\vec{S})$ are not aligned. We need to project the $\vec{\mu}_{J.}$ *onto the* $\vec{J}$. Consider the angle between $\vec{\mu}_{J.}$ *and* $\vec{J}$ *as* $\pi + \delta$.

Then, $|\overrightarrow{\mu_J}||\vec{J}|\cos(\pi + \delta) = \vec{\mu}_{J.} \cdot \vec{J}$ *and* $\cos\delta = -\frac{\vec{\mu}_{J.} \cdot \vec{J}}{|\overrightarrow{\mu_J}||\vec{J}|}$

$$|\overrightarrow{\mu_J}| = |\overrightarrow{\mu_J}|\cos\delta = |\overrightarrow{\mu_J}|\left(-\frac{\vec{\mu}_{J.} \cdot \vec{J}}{|\overrightarrow{\mu_J}||\vec{J}|}\right) = -\frac{\vec{\mu}_{J.} \cdot \vec{J}}{|\vec{J}|} = \frac{e}{2m}\frac{(\vec{L} + 2\vec{S}) \cdot (\vec{L} + \vec{S})}{|\vec{J}|}$$

$$= \frac{e}{2m}\frac{|\vec{L}|^2 + 2|\vec{S}|^2 + 3\vec{L} \cdot \vec{S}}{|\vec{J}|} =$$

$$since\ \vec{L} \cdot \vec{S} = \frac{1}{2}(|\vec{J}|^2 - |\vec{L}|^2 - |\vec{S}|^2)$$

$$|\overrightarrow{\mu_J}| = \frac{e}{2m}\frac{3|\vec{J}|^2 - |\vec{L}|^2 + |\vec{S}|^2}{2|\vec{J}|} = \frac{e}{2m}\frac{3j(j+1) - l(l+1) + s(s+1)}{2\sqrt{j(j+1)}}\hbar$$

Consider $\vec{B} = B_z\hat{z}$, then potential energy and since $cos\theta = \frac{J_z}{|\vec{J}|} = \frac{J_z}{\sqrt{j(j+1)}\hbar}$

$$U = -\vec{\mu}_{J.} \cdot \vec{B} = |\vec{\mu}_{J.}|B_z cos\theta = |\vec{\mu}_{J.}|B_z\frac{J_z}{\sqrt{j(j+1)}\hbar} = \frac{eJ_zB_z}{2m}\frac{3j(j+1) - l(l+1) + s(s+1)}{2j(j+1)} =$$

$$= g_{Laude}\frac{eJ_zB_z}{2m} \qquad = g_{Lande}\frac{eB_z}{2m}(m_j\hbar)$$

Where $g_{Lande} = \frac{3j(j+1) - l(l+1) + s(s+1)}{2j(j+1)}$

Note that

- if s=0 and j=l, $g_{Lande} = 1$
- If l=0 and j=s, $g_{Lande} = 2$
- Therefore, the degenerate E_n of the hydrogen atom is modified under a weak magnetic field as
- $E_{weak\ external\ magnetic\ field} = E_n + g_{Lande}\frac{eB_z}{2m}(m_j\hbar)$
- In this case, n, l, j, m_j are good quantum numbers

Then, what counts as a strong and weak external magnetic field? We use the interaction (L and S) between the magnetic field generated by the orbiting electron and the electron's spin.

$$U = -\vec{\mu}_{S.} \cdot \vec{B}_{due\ to\ \vec{L}}$$
$$\vec{B} = \frac{\mu_0 e}{4\pi m r^3}\vec{L}$$

For $2p\ state, l = 1, n = 2, and\ r = n^2 a_0$

$B = \frac{\mu_0 e}{4\pi m(4a_0)^3}\sqrt{1\cdot 2}\hbar$ = 0.28 T

Associated energy level $U = \mu_S\, B = 2 \text{ x } 10^{-5}\ eV$

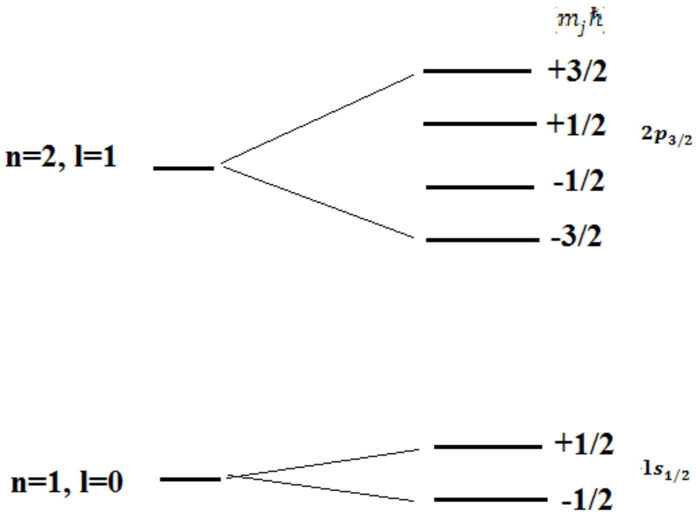
Therefore, the energy associated with LS coupling is very small.
A magnetic field that can interrupt the LS coupling, therefore, should be much larger than .28T.

Example : $2p_{3/2}$ →$1s_{1/2}$ in a weak external magnetic field of .05T

For $1s_{1/2,} l = 0, s = \frac{1}{2}, j = \frac{1}{2};\ m_j = -\frac{1}{2}, \frac{1}{2}$

For $2p_{3/2}, l = 1, s = \frac{1}{2}, j = \frac{3}{2}; m_j = -\frac{3}{2}, -\frac{1}{2}, +\frac{1}{2}, +\frac{3}{2}$

The split due to LS coupling is shown in the diagram**:**



We can calculate the amount of deviation occurs from the En value in each orbital by first calculating g_{Lande}

$$g_{Lande} = \frac{3j(j+1) - l(l+1) + s(s+1)}{2j(j+1)}$$

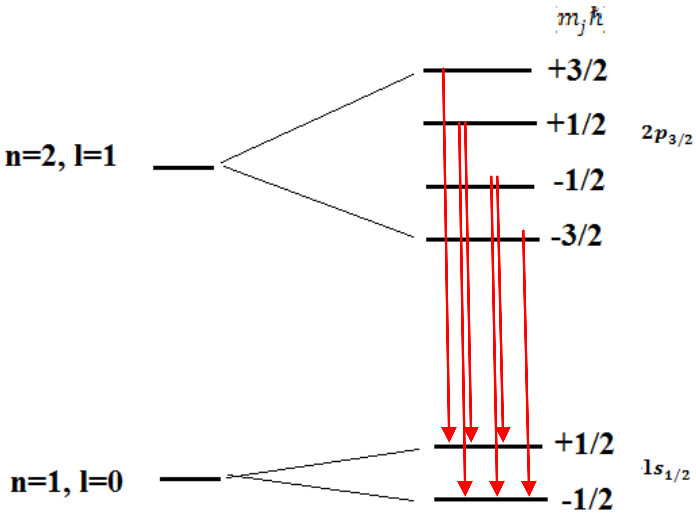
For $1s_{\frac{1}{2}}, l = 0, s = \frac{1}{2}, j = \frac{1}{2}$; $g_{Laude} = 2$;

$$\Delta E = g_{Lande} \frac{eB_z}{2m}(m_j\hbar) = \frac{2eB_z\hbar}{2m}(m_j) = \frac{2eB_z\hbar}{2m}\left(\pm\frac{1}{2}\right)$$

For $2p_{3/2}, l = 1, s = \frac{1}{2}, j = \frac{3}{2}$; $g_{Lande} = \frac{4}{3}$

$$\Delta E = g_{Lande} \frac{eB_z}{2m}(m_j\hbar) = \left(\frac{4}{3}\right)\frac{eB_z\hbar}{2m}(m_j) = \frac{2eB_z\hbar}{3m}\left(\pm\frac{3}{2}, \pm\frac{1}{2}\right)$$

Transition between the two states can occur when $\Delta m_j = 0, \pm 1$ (because photon's spin is 1 and the total angular momentum should be conserved before and after the transition).
This means that only the following transitions are allowed:



Therefore, a photon can have the energy difference:
$\Delta E = \Delta E_{without\ magnetic\ field} + \frac{2eB_z\hbar}{2m}\left(\pm\frac{5}{3}, \pm\frac{3}{3}, \pm\frac{1}{3}\right)$ → therefore 1 spectral line splits into 6 spectral lines in the presence of a weak magnetic field. The energy order of the split is 1.9 x $10^{-6} eV$. Compare this with the difference between $E_2 - E_1 = -13.6\left(\frac{1}{4} - 1\right)\ eV = 10.2\ eV$.