PH102: Interactive Lecture 3

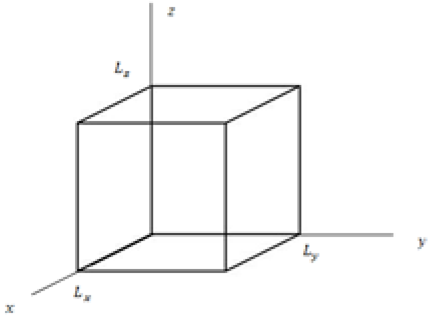
- Topics
  - Particle in a 3-D box
  - 3-D Schrodinger Equation for Hydrogen atom
  - $(x, y, z) \quad \leftrightarrow \quad (r, \theta, \phi)$
  - Separation of variables $R\Theta\Phi$
  - Three equations
  - Three quantum numbers (n, l, m_l)
  - Wave functions
  - Degeneracies

Particle in a 3-d infinite well

$$\frac{-\hbar^2}{2m}\nabla^2\psi(\vec{x}) + U(\vec{x})\psi(\vec{x}) = E\,\psi(\vec{x})$$

In (x, y, z) coordinates, $\vec{x} = (x, y, z)$, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi(x, y, z) = -\frac{2m}{\hbar^2}(E - U(x, y, z))\psi(x, y, z)$$



$$U(\vec{x}) = \begin{cases} 0 & 0 < x < L_x, 0 < y < L_y, 0 < z < L_z \\ \infty & otherwise \end{cases}$$

Wave functions exist only inside the 3-d infinite well.

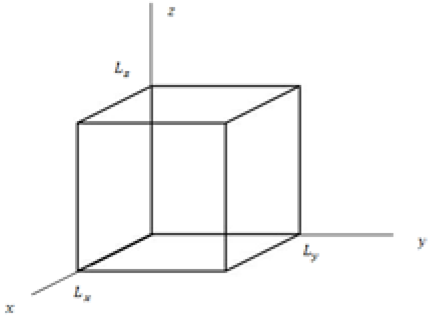
Separation of variables: $\psi(\vec{x}) = \psi(x, y, z) = F(x)G(y)H(z)$

Particle in a 3-d infinite well

$$\frac{-\hbar^2}{2m}\nabla^2\psi(\vec{x}) + U(\vec{x})\psi(\vec{x}) = E\,\psi(\vec{x})$$

In (x, y, z) coordinates, $\vec{x} = (x, y, z)$, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi(x, y, z) = -\frac{2m}{\hbar^2}(E - U(x, y, z))\psi(x, y, z)$$



$$U(\vec{x}) = \begin{cases} 0 & 0 < x < L_x, 0 < y < L_y, 0 < z < L_z \\ \infty & otherwise \end{cases}$$

Wave functions exist only inside the 3-d infinite well.

Separation of variables: $\psi(\vec{x}) = \psi(x, y, z) = F(x)G(y)H(z)$

$$\frac{1}{F(x)}\frac{\partial^2 F(x)}{\partial x^2} + \frac{1}{G(y)}\frac{\partial^2 G(y))}{\partial y^2} + \frac{1}{H(z)}\frac{\partial^2 H(z)}{\partial z^2} = -\frac{2mE}{\hbar^2}$$

=constant

constant=Cx **Constant=Cy** **Constant=Cz**

$$Cx + Cy + Cz = -\frac{2mE}{\hbar^2}$$

Particle in a 3-d infinite well

1-D solutions:

$$\frac{\partial^2 \psi(x)}{\partial x^2} = \frac{-2m\,E}{\hbar^2}\,\psi(x)$$

Wave function: $\psi_{0<x<L}(x) = \sqrt{\frac{2}{L}}\ \sin(\frac{n\pi x}{L})$

Energy $E = \frac{n^2\,\pi^2\,\hbar^2}{2mL^2}$

3-D solutions: wave functions

$$\begin{cases} \frac{d\,F(x)}{dx^2} = C_x F(x) & \rightarrow F(x) = A_x \sin\frac{n_x \pi x}{L_x} \\ \frac{d\,G(y)}{dy^2} = C_y G(y) & \rightarrow G(y) = A_y \sin\frac{n_y \pi y}{L_y} \\ \frac{d\,H(z)}{dz^2} = C_z H(z) & \rightarrow H(z) = A_z \sin\frac{n_z \pi z}{L_z} \end{cases}$$

Particle in a 3-d infinite well

1-D solutions:

$$\frac{\partial^2 \psi(x)}{\partial x^2} = \frac{-2m\,E}{\hbar^2}\,\psi(x)$$

Wave function: $\psi_{0<x<L}(x) = \sqrt{\frac{2}{L}}\,\sin(\frac{n\pi x}{L})$

Energy $E = \frac{n^2\,\pi^2\,\hbar^2}{2mL^2}$

3-D solutions: wave functions

$$\begin{cases} \frac{d\,F(x)}{dx^2} = C_x F(x) & \rightarrow F(x) = A_x \sin\frac{n_x \pi x}{L_x} \\ \frac{d\,G(y)}{dy^2} = C_y G(y) & \rightarrow G(y) = A_y \sin\frac{n_y \pi y}{L_y} \\ \frac{d\,H(z)}{dz^2} = C_z H(z) & \rightarrow H(z) = A_z \sin\frac{n_z \pi z}{L_z} \end{cases}$$

Cx = $-\frac{n_x^2\pi^2}{L_x{}^2}$ Cy = $-\frac{n_y^2\pi^2}{L_y{}^2}$ Cz= $-\frac{n_z^2\pi^2}{L_z{}^2}$

Cx + Cy + Cz = $-\frac{2mE}{\hbar^2}$ = $-\frac{n_x^2\pi^2}{L_x{}^2} - \frac{n_y^2\pi^2}{L_y{}^2} - \frac{n_z^2\pi^2}{L_z{}^2}$

Particle in a 3-D infinite well

$$E_{(n_x,n_y,n_z)} = \left(\frac{n_x^2}{L_x^{\ 2}} + \frac{n_y^2}{L_y^{\ 2}} + \frac{n_z^2}{L_z^{\ 2}}\right)\frac{\pi^2\hbar^2}{2m}$$

$$\psi\,(x,y,z) = F(x)G(y)H(z) = A\ \sin\frac{n_x\pi x}{L_x}\sin\frac{n_y\pi y}{L_y}\sin\frac{n_z\pi z}{L_z}$$

Lowest energy state = $E_{(1,1,1)} = \left(\frac{1^2}{L_x^{\ 2}} + \frac{1^2}{L_y^{\ 2}} + \frac{1^2}{L_z^{\ 2}}\right)\frac{\pi^2\hbar^2}{2m}$

Wave function for the lowest energy state = $\psi_{(1,1,1)} = A\ \sin\frac{\pi x}{L_x}\sin\frac{\pi y}{L_y}\sin\frac{\pi z}{L_z}$

,

$$E_{(n_x,n_y,n_z)} = \left(\frac{n_x^2}{L_x{}^2} + \frac{n_y^2}{L_y{}^2} + \frac{n_z^2}{L_z{}^2}\right)\frac{\pi^2\hbar^2}{2m}$$

$$\psi\,(x,y,z) = F(x)G(y)H(z) = A\,\sin\frac{n_x\pi x}{L_x}\sin\frac{n_y\pi y}{L_y}\sin\frac{n_z\pi z}{L_z}$$

An electron in a cubit 3d infinite well of 1 nm at the E(2,1,1) state

$$E_{(2,1,1)} = 6\left(\frac{\pi^2\hbar^2}{2mL^2}\right) = (2^2 + 1^2 + 1^2)\frac{\pi^2\,(1.055 \text{ x } 10^{-34}\text{ J sec})^2}{2(9.11 \text{ x } 10^{-31}\text{kg})(10^{-9}\text{ m})^2}$$

$$= 3.62 \text{ x } 10^{-19}\text{ J} = 2.26 \text{ eV} \text{ (the same as } E_{(1,2,1)} = E_{(1,1,2)})$$

$$\text{Where}\begin{cases}\text{electron mass} = 9.11 \text{ x } 10^{-31}\text{kg}\\ h = 1.055 \text{ x } 10^{-34}\text{ J sec}\\ L = 10^{-9}\text{ m}\end{cases} \quad \text{and } 1 \text{ eV} = 1.602 \text{ x } 10^{-19}\text{ J}$$

,

$$E_{(n_x,n_y,n_z)} = \left(\frac{n_x^2}{L_x{}^2} + \frac{n_y^2}{L_y{}^2} + \frac{n_z^2}{L_z{}^2}\right)\frac{\pi^2\hbar^2}{2m}$$

$$\psi(x,y,z) = F(x)G(y)H(z) = A \sin\frac{n_x\pi x}{L_x}\sin\frac{n_y\pi y}{L_y}\sin\frac{n_z\pi z}{L_z}$$

Where is a particle with the value most likely to be found?

Probability density

- $|\psi_{(2,1,1)}|^2 = A^2\left(\sin\frac{2\pi x}{L}\right)^2\left(\sin\frac{\pi y}{L}\right)^2\left(\sin\frac{\pi z}{L}\right)^2$

,

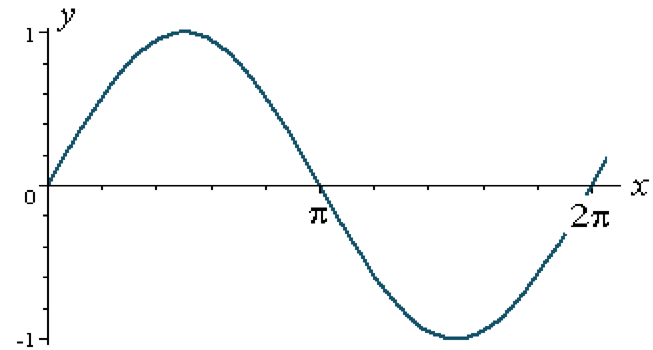
$$E_{(n_x,n_y,n_z)} = \left(\frac{n_x^2}{L_x^{\,2}} + \frac{n_y^2}{L_y^{\,2}} + \frac{n_z^2}{L_z^{\,2}}\right)\frac{\pi^2\hbar^2}{2m}$$

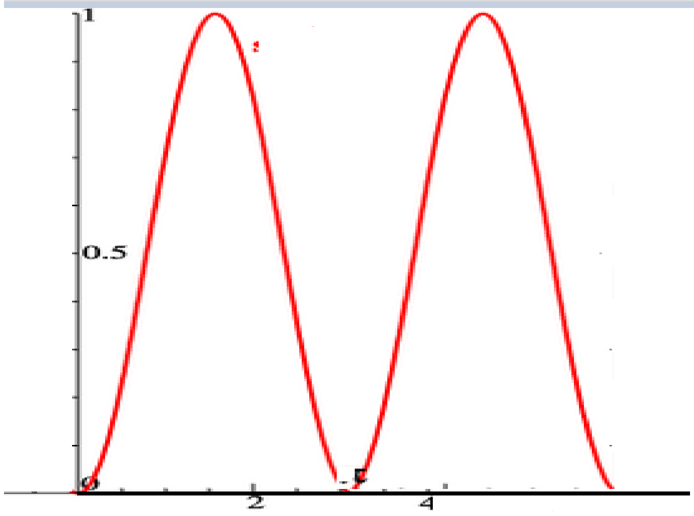
$$\psi\,(x,y,z) = F(x)G(y)H(z) = A\,\sin\frac{n_x\pi x}{L_x}\sin\frac{n_y\pi y}{L_y}\sin\frac{n_z\pi z}{L_z}$$

Where is a particle with the value most likely to be found?

Probability density

- $|\psi_{(2,1,1)}|^2 = A^2\,(\sin\frac{2\pi x}{L})^2\,(\sin\frac{\pi y}{L})^2\,(\sin\frac{\pi z}{L})^2$

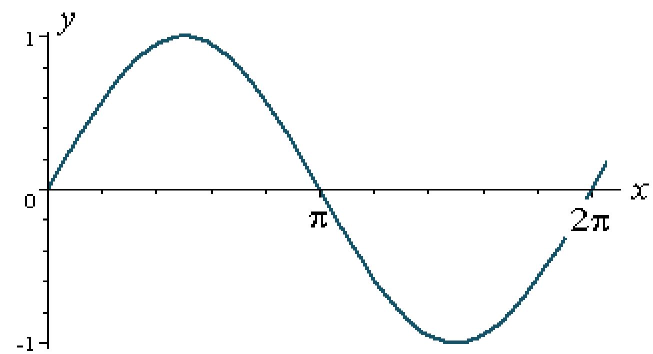




$$E_{(n_x,n_y,n_z)} = \left(\frac{n_x^2}{{L_x}^2} + \frac{n_y^2}{{L_y}^2} + \frac{n_z^2}{{L_z}^2}\right)\frac{\pi^2\hbar^2}{2m}$$

$$\psi(x,y,z) = F(x)G(y)H(z) = A\sin\frac{n_x\pi x}{L_x}\sin\frac{n_y\pi y}{L_y}\sin\frac{n_z\pi z}{L_z}$$

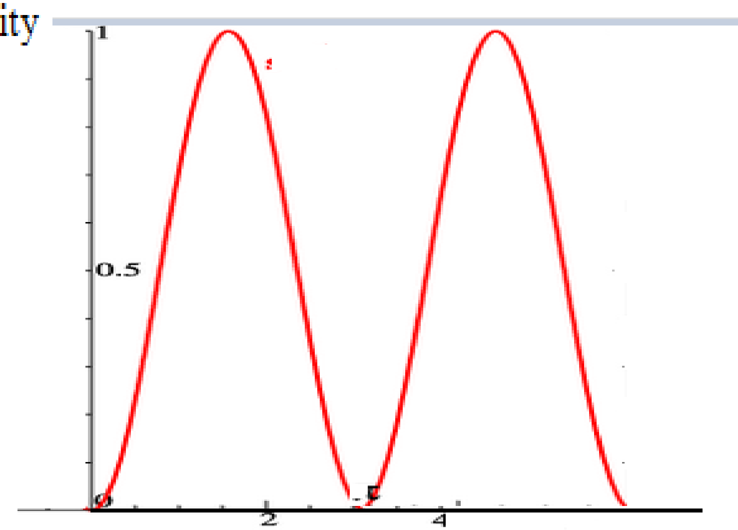
Where is a particle with the value most likely to be found?



Probability density

- $|\psi_{(2,1,1)}|^2 = A^2\left(\sin\frac{2\pi x}{L}\right)^2\left(\sin\frac{\pi y}{L}\right)^2\left(\sin\frac{\pi z}{L}\right)^2$

Since the value of $(sin\theta)^2$ *is highest when* $\theta = \frac{1}{2}\pi, \frac{3}{2}\pi, etc.$, the probability



$$\begin{cases} x = \frac{L}{4}, \frac{3L}{4} \\ y = \frac{L}{2} \\ z = \frac{L}{2} \end{cases}$$

Energy Split

Consider (that is, a slightly non-symmetric box along the z axis)

$L_x = L_y = L_z = L$ → $L_x = L_y = L,\ L_z = .9\,L$

$$E_{(1,1,1)} = (1^2 + 1^2 + 1^2)\left(\frac{\pi^2\hbar^2}{2mL^2}\right) = 3\left(\frac{\pi^2\hbar^2}{2mL^2}\right)$$

$$E_{(2,1,1)} = E_{(1,2,1)} = E_{(1,1,2)} = (2^2 + 1^2 + 1^2)\left(\frac{\pi^2\hbar^2}{2mL^2}\right) = 6\left(\frac{\pi^2\hbar^2}{2mL^2}\right)$$

Energy Split

Consider (that is, a slightly non-symmetric box along the z axis)

$$L_x = L_y = L_z = L \longrightarrow L_x = L_y = L,\ L_z = .9\,L$$

$$E_{(1,1,1)} = (2^2 + 1^2 + 1^2)\left(\frac{\pi^2\hbar^2}{2mL^2}\right) = 3\,\left(\frac{\pi^2\hbar^2}{2mL^2}\right)$$

$$E_{(1,1,1)} = \left(\frac{1^2}{L^2} + \frac{1^2}{L^2} + \frac{1^2}{.9^2L^2}\right)\left(\frac{\pi^2\hbar^2}{2m}\right) = (1 + 1 + 1.23)\left(\frac{\pi^2\hbar^2}{2mL^2}\right) = 3.23\left(\frac{\pi^2\hbar^2}{2mL^2}\right)$$

$$E_{(2,1,1)} = E_{(1,2,1)} = E_{(1,1,2)} = (2^2 + 1^2 + 1^2)\left(\frac{\pi^2\hbar^2}{2mL^2}\right) = 6\left(\frac{\pi^2\hbar^2}{2mL^2}\right)$$

$$E_{(2,1,1)} = E_{(1,2,1)} = \left(\frac{2^2}{L^2} + \frac{1^2}{L^2} + \frac{1^2}{.9^2L^2}\right)\left(\frac{\pi^2\hbar^2}{2m}\right) = (4 + 1 + 1.23)\left(\frac{\pi^2\hbar^2}{2mL^2}\right) = 6.23\left(\frac{\pi^2\hbar^2}{2mL^2}\right)$$

$$E_{(1,1,2)} = \left(\frac{1^2}{L^2} + \frac{1^2}{L^2} + \frac{2^2}{.9^2L^2}\right)\left(\frac{\pi^2\hbar^2}{2m}\right) = (1 + 1 + 4.92)\left(\frac{\pi^2\hbar^2}{2mL^2}\right) = 6.92\left(\frac{\pi^2\hbar^2}{2mL^2}\right)$$

Energy Split

Consider (that is, a slightly non-symmetric box along the z axis)

$$L_x = L_y = L_z = L \longrightarrow L_x = L_y = L,\ L_z = .9\,L$$

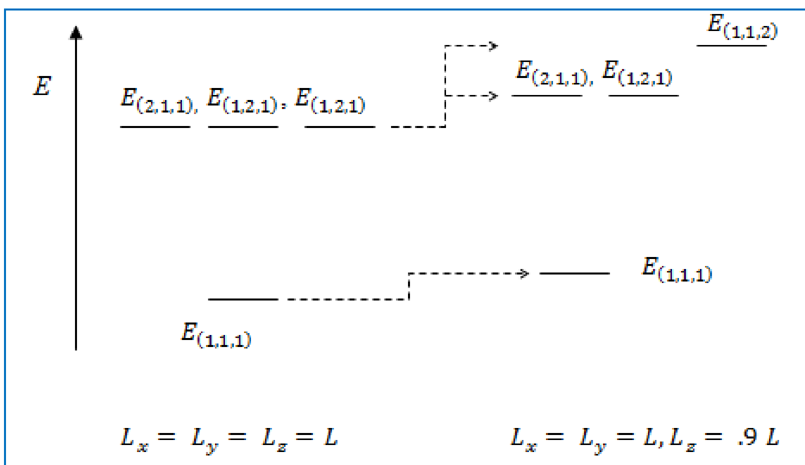
$$E_{(1,1,1)} = (2^2 + 1^2 + 1^2)\left(\frac{\pi^2\hbar^2}{2mL^2}\right) = 3\left(\frac{\pi^2\hbar^2}{2mL^2}\right)$$

$$E_{(1,1,1)} = \left(\frac{1^2}{L^2} + \frac{1^2}{L^2} + \frac{1^2}{.9^2L^2}\right)\left(\frac{\pi^2\hbar^2}{2m}\right) = (1 + 1 + 1.23)\left(\frac{\pi^2\hbar^2}{2mL^2}\right) = 3.23\left(\frac{\pi^2\hbar^2}{2mL^2}\right)$$

$$E_{(2,1,1)} = E_{(1,2,1)} = E_{(1,1,2)} = (2^2 + 1^2 + 1^2)\left(\frac{\pi^2\hbar^2}{2mL^2}\right) = 6\left(\frac{\pi^2\hbar^2}{2mL^2}\right)$$

$$) = \left(\frac{2^2}{L^2} + \frac{1^2}{L^2} + \frac{1^2}{.9^2L^2}\right)\left(\frac{\pi^2\hbar^2}{2m}\right) = (4 + 1 + 1.23)\left(\frac{\pi^2\hbar^2}{2mL^2}\right) = 6.23\left(\frac{\pi^2\hbar^2}{2mL^2}\right)$$

$$\frac{1^2}{L^2} + \frac{2^2}{.9^2L^2}\left)\left(\frac{\pi^2\hbar^2}{2m}\right) = (1 + 1 + 4.92)\left(\frac{\pi^2\hbar^2}{2mL^2}\right) = 6.92\left(\frac{\pi^2\hbar^2}{2mL^2}\right)\right.$$



Quantum Problem Solving Schema

- Think about potential energy and Hamiltonian
- Divide regions, if possible
- Choose appropriate coordinates
- Write Schrodinger Equation for each region
- Separation of variables, if possible
- Set "constant"s if applicable
- Solve for wave function
  - Boundary conditions
  - Normalization
- Solve for energy
- Build energy level diagrams and inspect for energy degeneracies

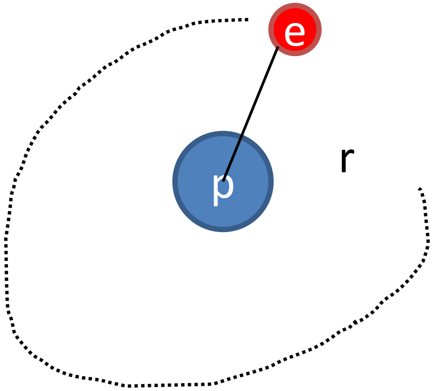
Hydrogen atom

- Potential created by Coulomb interactions between electron ($-e$) and proton ($+e$)

Hydrogen atom

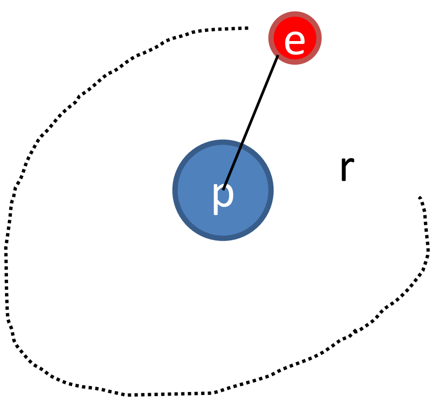
- Potential created by Coulomb interactions between electron (-*e*) and proton (+*e*)

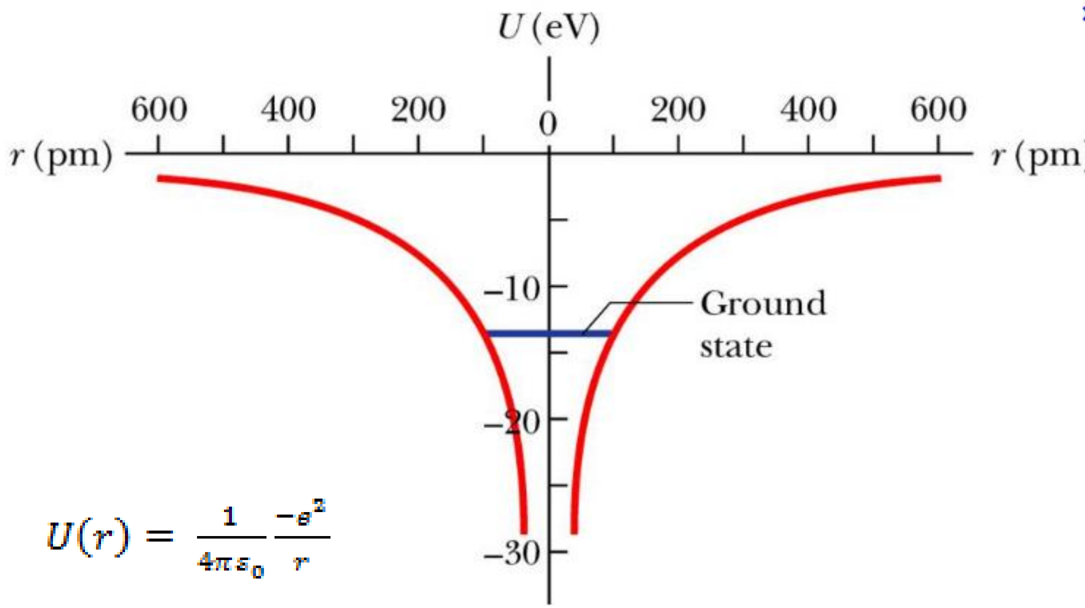
$$U(r) = \frac{1}{4\pi\varepsilon_0}\frac{-e^2}{r}$$



Hydrogen atom

- Potential created by Coulomb interactions between electron (-*e*) and proton (+*e*)

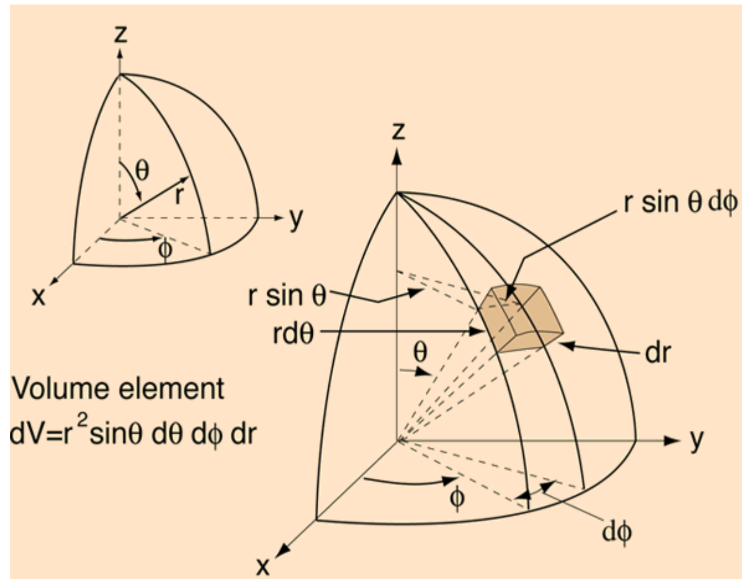




$$U(r) = \frac{1}{4\pi\varepsilon_0}\frac{-e^2}{r}$$

$$(x, y, z) \quad \leftrightarrow \quad (r, \theta, \phi)$$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \phi = \tan^{-1}\dfrac{y}{x} \\ \theta = \cos^{-1}\dfrac{z}{r} \end{cases}$$



$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla^2 = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \csc\theta\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + \csc^2\theta\,\frac{\partial}{\partial\phi^2}\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi^2}$$

Schrodinger Equation

$$\frac{-\hbar^2}{2m}\nabla^2 \psi(\vec{x}) + U(\vec{x})\psi(\vec{x}) = E\,\psi(\vec{x})$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial}{\partial\phi^2}$$

Schrodinger Equation

$$\frac{-\hbar^2}{2m} \nabla^2 \psi\,(\vec{x}) + U(\vec{x})\psi\,(\vec{x}) = E\;\psi(\vec{x})$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial}{\partial\phi^2}$$

$$\frac{-\hbar^2}{2m}\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi(r,\theta,\phi) + U(r)\,\psi(r,\theta,\phi) = E\;\psi(r,\theta,\phi)$$

Schrodinger Equation

$$\frac{-\hbar^2}{2m}\nabla^2\psi(\vec{x}) + U(\vec{x})\psi(\vec{x}) = E\,\psi(\vec{x})$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial}{\partial\phi^2}$$

$$\frac{-\hbar^2}{2m}\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi(r,\theta,\phi) + U(r)\,\psi(r,\theta,\phi) = E\,\psi(r,\theta,\phi)$$

$$\frac{-\hbar^2}{2m}\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi = (E-U)\,\psi$$

Schrodinger Equation

$$\frac{-\hbar^2}{2m}\nabla^2\psi(\vec{x}) + U(\vec{x})\psi(\vec{x}) = E\,\psi(\vec{x})$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial}{\partial\phi^2}$$

$$\frac{-\hbar^2}{2m}\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi(r,\theta,\phi) + U(r)\,\psi(r,\theta,\phi) = E\,\psi(r,\theta,\phi)$$

$$\frac{-\hbar^2}{2m}\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi = (E-U)\,\psi$$

$$\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi = -\frac{2mr^2}{\hbar^2}\,(E-U)\,\psi$$

Schrodinger Equation

$$\frac{-\hbar^2}{2m}\nabla^2\psi(\vec{x}) + U(\vec{x})\psi(\vec{x}) = E\,\psi(\vec{x})$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial}{\partial\phi^2}$$

$$\frac{-\hbar^2}{2m}\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi(r,\theta,\phi) + U(r)\,\psi(r,\theta,\phi) = E\,\psi(r,\theta,\phi)$$

$$\frac{-\hbar^2}{2m}\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi = (E-U)\,\psi$$

$$\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi = -\frac{2mr^2}{\hbar^2}(E-U)\,\psi$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\psi + \frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\psi\right) + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\,\psi = -\frac{2mr^2}{\hbar^2}(E-U)\,\psi$$

Schrodinger Equation

$$\frac{-\hbar^2}{2m}\nabla^2\psi(\vec{x})+U(\vec{x})\psi(\vec{x})=E\,\psi(\vec{x})$$

$$\nabla^2=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2 sin^2\theta}\frac{\partial}{\partial\phi^2}$$

$$\frac{-\hbar^2}{2m}\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi(r,\theta,\phi)+U(r)\,\psi(r,\theta,\phi)=E\,\psi(r,\theta,\phi)$$

$$\frac{-\hbar^2}{2m}\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi=(E-U)\,\psi$$

$$\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi=-\frac{2mr^2}{\hbar^2}(E-U)\,\psi$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\psi+\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\psi\right)+\frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi=-\frac{2mr^2}{\hbar^2}(E-U)\,\psi$$

$$\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)\psi+\frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi=\left[-\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-\frac{2mr^2}{\hbar^2}(E-U)\right]\psi$$

Schrodinger Equation

$$\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)\psi + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi = \left[-\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{2mr^2}{\hbar^2}(E-U)\right]\psi$$

Separation of variables

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$$

Schrodinger Equation

$$\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)\psi + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi = \left[-\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{2mr^2}{\hbar^2}(E-U)\right]\psi$$

Separation of variables

$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$

$$\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)R\Theta\Phi + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}R\Theta\Phi = \left[-\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{2mr^2}{\hbar^2}(E-U)\right]R\Theta\Phi$$

$$\frac{\partial\psi}{\partial r} = \Theta\Phi\frac{\partial R}{\partial r}$$

$$\frac{\partial\psi}{\partial\theta} = R\Phi\frac{\partial\Theta}{\partial\theta}$$

$$\frac{\partial^2\Phi}{\partial\phi^2} = R\Theta\frac{\partial^2\Phi}{\partial\phi^2}$$

Schrodinger Equation

$$\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)\psi + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi = \left[-\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{2mr^2}{\hbar^2}(E-U)\right]\psi$$

Separation of variables

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$$

$$\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)R\Theta\Phi + \frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2}R\Theta\Phi = \left[-\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{2mr^2}{\hbar^2}(E-U)\right]R\Theta\Phi$$

$$\frac{\partial\psi}{\partial r} = \Theta\Phi\frac{\partial R}{\partial r}$$

$$\frac{\partial\psi}{\partial\theta} = R\Phi\frac{\partial\Theta}{\partial\theta}$$

$$\frac{\partial^2\Phi}{\partial\phi^2} = R\Theta\frac{\partial^2\Phi}{\partial\phi^2}$$

$$R\Phi\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\Theta}{\partial\theta}\right) + R\Theta\frac{1}{sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = -\Theta\Phi\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - \frac{2mr^2}{\hbar^2}(E-U)R\Theta\Phi$$

Schrodinger Equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\psi + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\psi = \left[-\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{2mr^2}{\hbar^2}(E-U)\right]\psi$$

Separation of variables

$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)R\Theta\Phi + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}R\Theta\Phi = \left[-\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{2mr^2}{\hbar^2}(E-U)\right]R\Theta\Phi$$

$$\frac{\partial\psi}{\partial r} = \Theta\Phi\frac{\partial R}{\partial r}$$

$$\frac{\partial\psi}{\partial\theta} = R\Phi\frac{\partial\Theta}{\partial\theta}$$

$$\frac{\partial^2\Phi}{\partial\phi^2} = R\Theta\frac{\partial^2\Phi}{\partial\phi^2}$$

$$R\Phi\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + R\Theta\frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = -\Theta\Phi\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - \frac{2mr^2}{\hbar^2}(E-U)R\Theta\Phi$$

$$\frac{1}{\Theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\Phi}\frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = -\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - \frac{2mr^2}{\hbar^2}(E-U)$$

Schrodinger Equation

$$\frac{1}{\Theta}\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Phi}\frac{1}{sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=-\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)-\frac{2mr^2}{\hbar^2}(E-U)=C\ (\text{Constant})$$

-

-l (l + 1)

Angular part

Radial part

Schrodinger Equation

$$\frac{1}{\Theta}\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Phi}\frac{1}{sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=-\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)-\frac{2mr^2}{\hbar^2}(E-U)=C\ (\text{Constant})$$

-

-l (l + 1)

Angular part

Radial part

-l (l + 1)

-l (l + 1)

Schrodinger Equation

$$\frac{1}{\Theta}\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Phi}\frac{1}{sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=-\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)-\frac{2mr^2}{\hbar^2}(E-U)=C \text{ (Constant)}$$

-

-l (l + 1)

Angular part ***-l (l + 1)***

Radial part ***-l (l + 1)***

$$\begin{cases} -\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)-\frac{2mr^2}{\hbar^2}\left(E-U(r)\right)=C=-l(l+1) \text{ -} \\ \frac{1}{\Theta}\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Phi}\frac{1}{sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=C=-l(l+1) \end{cases}$$

Schrodinger Equation

$$\frac{1}{\Theta}\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Phi}\frac{1}{sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=-\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)-\frac{2mr^2}{\hbar^2}(E-U)=C\ (\text{Constant})$$

-l (l + 1)

Angular part -l (l + 1)

Radial part -l (l + 1)

$$\begin{cases} -\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)-\frac{2mr^2}{\hbar^2}\left(E-U(r)\right)=C=-l(l+1) \\ \frac{1}{\Theta}\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Phi}\frac{1}{sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=C=-l(l+1) \end{cases}$$

$multiply\ sin^2\theta$.

Schrodinger Equation

$$\frac{1}{\Theta}\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Phi}\frac{1}{sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=-\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)-\frac{2mr^2}{\hbar^2}(E-U)=C\ (\text{Constant})$$

-l (l + 1)

Angular part -l (l + 1)

Radial part -l (l + 1)

$$\begin{cases} -\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)-\frac{2mr^2}{\hbar^2}(E-U(r))=C=-l(l+1) \\ \frac{1}{\Theta}\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Phi}\frac{1}{sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=C=-l(l+1) \end{cases}$$

$$multiply\ sin^2\theta. \quad \frac{1}{\Theta}sin\theta\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2}=-l(l+1)\ sin^2\theta$$

Schrodinger Equation

$$\frac{1}{\Theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Phi}\frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=-\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)-\frac{2mr^2}{\hbar^2}(E-U)=C\ (\text{Constant})$$

-l (l + 1)

Angular part -l (l + 1)

Radial part -l (l + 1)

$$\begin{cases}-\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)-\frac{2mr^2}{\hbar^2}\left(E-U(r)\right)=C=-l(l+1)\\ \frac{1}{\Theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Phi}\frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=C=-l(l+1)\end{cases}$$

multiply $\sin^2\theta$. $\quad \frac{1}{\Theta}\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2}=-l(l+1)\sin^2\theta$

$$\frac{1}{\Theta}\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+l(l+1)\sin^2\theta=-\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2}$$

$$=m_l^2\ (\text{another constant})$$

Schrodinger Equation

$$\begin{cases} \dfrac{\partial^2 \Phi}{\partial \phi^2} = -m_l^2 \Phi \\ sin\theta \dfrac{\partial}{\partial \theta}\left(sin\theta \dfrac{\partial \Theta}{\partial \theta}\right) + [l(l+1)sin^2\theta - m_l^2]\Theta = 0 \\ \dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial R}{\partial r}\right) + \dfrac{2mr^2}{\hbar^2}\left(E - U(r)\right)R - l(l+1)R = 0 \end{cases}$$

Azimuthal Equation

Polar Equation

Radial Equation

Azimuthal Equation

$$\frac{\partial^2 \Phi}{\partial \phi^2} = -m_l^2 \Phi$$

Wave function

Boundary condition

Quantization

Azimuthal Equation

$$\frac{\partial^2 \Phi}{\partial \phi^2} = -m_l^2 \Phi$$

Wave function $\Phi(\phi) = A\, e^{i m_l \phi}$

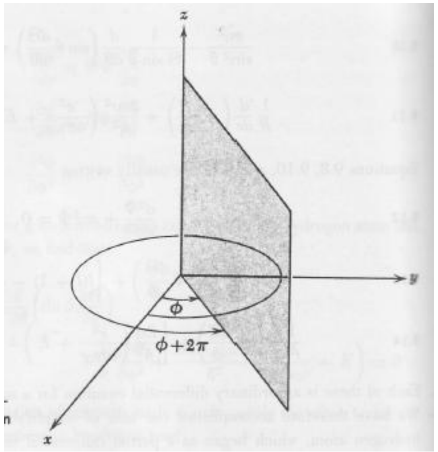
Boundary condition

Quantization

Azimuthal Equation

$$\frac{\partial^2 \Phi}{\partial \phi^2} = -m_l^2 \Phi$$

Wave function $\Phi(\phi) = A\, e^{i m_l \phi}$



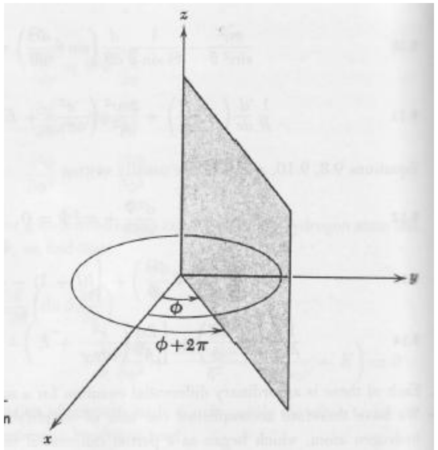
Boundary condition

Quantization

Azimuthal Equation

$$\frac{\partial^2 \Phi}{\partial \phi^2} = -m_l^2 \Phi$$

Wave function $$\Phi(\phi) = A\, e^{im_l \phi}$$



Boundary condition

$$\Phi(\phi) = \Phi(\phi + 2\pi)$$

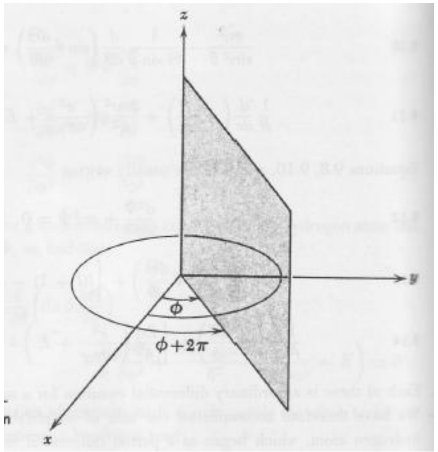
$$A\, e^{im_l \phi} = A\, e^{im_l(\phi+2\pi)} = A e^{im_l \phi} e^{i2\pi m_l}$$

Quantization

Azimuthal Equation

$$\frac{\partial^2 \Phi}{\partial \phi^2} = -m_l^2 \Phi$$

Wave function $\Phi(\phi) = A\, e^{i m_l \phi}$



Boundary condition

$$\Phi(\phi) = \Phi(\phi + 2\pi)$$

$$A\, e^{i m_l \phi} = A\, e^{i m_l (\phi + 2\pi)} = A e^{i m_l \phi} e^{i 2\pi m_l}$$

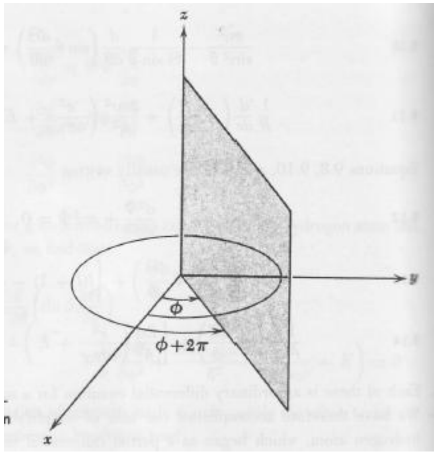
$$e^{i 2\pi m_l} = 1 = cos 2\pi m_l + \; i sin 2\pi m_l$$

Quantization

Azimuthal Equation

$$\frac{\partial^2 \Phi}{\partial \phi^2} = -m_l^2 \Phi$$

Wave function $\Phi(\phi) = A\, e^{i m_l \phi}$



Boundary condition

$$\Phi(\phi) = \Phi(\phi + 2\pi)$$

$$A\, e^{i m_l \phi} = A\, e^{i m_l (\phi + 2\pi)} = A e^{i m_l \phi} e^{i 2\pi m_l}$$

$$e^{i 2\pi m_l} = 1 = cos 2\pi m_l + \; i sin 2\pi m_l$$

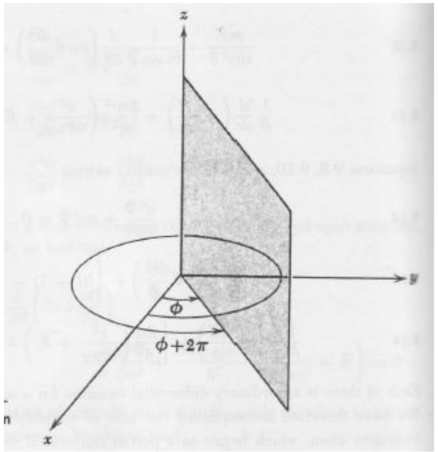
Quantization

$$m_l = 0, \pm 1, \pm 2, \pm 3, etc.$$

Azimuthal Equation

$$\frac{\partial^2 \Phi}{\partial \phi^2} = -m_l^2 \Phi$$

Wave function $\Phi(\phi) = A\, e^{im_l \phi}$



Boundary condition

$$\Phi(\phi) = \Phi(\phi + 2\pi)$$

$$A\, e^{im_l \phi} = A\, e^{im_l(\phi + 2\pi)} = A e^{im_l \phi} e^{i2\pi m_l}$$

$$e^{i2\pi m_l} = 1 = cos 2\pi m_l + isin 2\pi m_l$$

Quantization

$$m_l = 0, \pm 1, \pm 2, \pm 3, etc.$$

Magnetic quantum number

Polar Equation

$$sin\theta \frac{\partial}{\partial\theta}\left(sin\theta \frac{\partial\Theta}{\partial\theta}\right) + [l(l+1)sin^2\theta - m_l^2]\Theta = 0$$

Solutions: Associated Legendre Functions

Quantization

any given l, m_l $values\ can\ be$ $0, \pm 1, \pm 2, \pm l$

→ **Orbital quantum number**

n	l	m_l	$\Phi(\phi)$	$\Theta(\theta)$	$R(r)$	$\psi\,(r,\theta,\phi)$
1	0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2}}$	$\frac{2}{a_0{}^{3/2}}\, e^{-r/a_0}$	$\frac{1}{\sqrt{\pi}\, a_0{}^{3/2}}\, e^{-r/a_0}$
2	0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2\sqrt{2}\, a_0{}^{3/2}}\left(2-\frac{r}{a_0}\right) e^{-r/2a_0}$	$\frac{1}{4\sqrt{2\pi}\, a_0{}^{3/2}}\left(2-\frac{r}{a_0}\right) e^{-r/2a_0}$
2	1	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{\sqrt{6}}{2}\cos\theta$	$\frac{1}{2\sqrt{6}\, a_0{}^{3/2}}\frac{r}{a_0}\, e^{-r/2a_0}$	$\frac{1}{4\sqrt{2\pi}\, a_0{}^{3/2}}\frac{r}{a_0}\, e^{-r/2a_0}\cos\theta$
2	1	± 1	$\frac{1}{\sqrt{2\pi}}\, e^{\pm i\phi}$	$\frac{\sqrt{3}}{2}\sin\theta$	$\frac{1}{2\sqrt{6}\, a_0{}^{3/2}}\frac{r}{a_0}\, e^{-r/2a_0}$	$\frac{1}{8\sqrt{\pi}\, a_0{}^{3/2}}\frac{r}{a_0}\, e^{-r/2a_0}\sin\theta\, e^{\pm i\phi}$
3	0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2}}$	$\frac{2}{81\sqrt{3}\, a_0{}^{3/2}}\left(27-18\frac{r}{a_0}+2\frac{r^2}{a_0{}^2}\right) e^{-r/3a_0}$	$\frac{1}{81\sqrt{3\pi}\, a_0{}^{3/2}}\left(27-18\frac{r}{a_0}+2\frac{r^2}{a_0{}^2}\right) e^{-r/3a_0}$
3	1	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{\sqrt{6}}{2}\cos\theta$	$\frac{4}{81\sqrt{6}\, a_0{}^{3/2}}\left(6-\frac{r}{a_0}\right)\frac{r}{a_0}\, e^{-r/3a_0}$	$\frac{\sqrt{2}}{81\sqrt{\pi}\, a_0{}^{3/2}}\left(6-\frac{r}{a_0}\right)\frac{r}{a_0}\, e^{-r/3a_0}\cos\theta$
3	1	± 1	$\frac{1}{\sqrt{2\pi}}\, e^{\pm i\phi}$	$\frac{\sqrt{3}}{2}\sin\theta$	$\frac{4}{81\sqrt{6}\, a_0{}^{3/2}}\left(6-\frac{r}{a_0}\right)\frac{r}{a_0}\, e^{-r/3a_0}$	$\frac{1}{81\sqrt{\pi}\, a_0{}^{3/2}}\left(6-\frac{r}{a_0}\right)\frac{r}{a_0}\, e^{-r/3a_0}\sin\theta\, e^{\pm i\phi}$
3	2	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{\sqrt{10}}{4}(3\cos^2\theta-1)$	$\frac{4}{81\sqrt{30}\, a_0{}^{3/2}}\frac{r^2}{a_0{}^2}\, e^{-r/3a_0}$	$\frac{1}{81\sqrt{6\pi}\, a_0{}^{3/2}}\frac{r^2}{a_0{}^2}\, e^{-r/3a_0}(3\cos^2\theta-1)$
3	2	± 1	$\frac{1}{\sqrt{2\pi}}\, e^{\pm i\phi}$	$\frac{\sqrt{15}}{2}\sin\theta\cos\theta$	$\frac{4}{81\sqrt{30}\, a_0{}^{3/2}}\frac{r^2}{a_0{}^2}\, e^{-r/3a_0}$	$\frac{1}{81\sqrt{\pi}\, a_0{}^{3/2}}\frac{r^2}{a_0{}^2}\, e^{-r/3a_0}\sin\theta\cos\theta\, e^{\pm i\phi}$
3	2	± 2	$\frac{1}{\sqrt{2\pi}}\, e^{\pm 2i\phi}$	$\frac{\sqrt{15}}{4}\sin^2\theta$	$\frac{4}{81\sqrt{30}\, a_0{}^{3/2}}\frac{r^2}{a_0{}^2}\, e^{-r/3a_0}$	$\frac{1}{162\sqrt{\pi}\, a_0{}^{3/2}}\frac{r^2}{a_0{}^2}\, e^{-r/3a_0}\sin^2\theta\, e^{\pm 2i\phi}$

Radial Equation

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{2mr^2}{\hbar^2}(E - U(r))R - l(l+1)R = 0$$

Solutions: associated Laguerre functions
Quantization:

$$E_n = -\frac{me^4}{32\pi \varepsilon_0^2 \hbar^2}\left(\frac{1}{n^2}\right) \text{ where } n \text{ is an integer}$$

n → **Principal quantum number**

l = 0, 1, 2, …(*n*-1)

n	l	m_l	$\Phi(\phi)$	$\Theta(\theta)$	$R(r)$	$\psi(r,\theta,\phi)$
1	0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2}}$	$\frac{2}{a_0{}^{3/2}} e^{-r/a_0}$	$\frac{1}{\sqrt{\pi}\, a_0{}^{3/2}} e^{-r/a_0}$
2	0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2\sqrt{2}\, a_0{}^{3/2}}\left(2-\frac{r}{a_0}\right) e^{-r/2a_0}$	$\frac{1}{4\sqrt{2\pi}\, a_0{}^{3/2}}\left(2-\frac{r}{a_0}\right) e^{-r/2a_0}$
2	1	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{\sqrt{6}}{2}\cos\theta$	$\frac{1}{2\sqrt{6}\, a_0{}^{3/2}}\frac{r}{a_0} e^{-r/2a_0}$	$\frac{1}{4\sqrt{2\pi}\, a_0{}^{3/2}}\frac{r}{a_0} e^{-r/2a_0}\cos\theta$
2	1	±1	$\frac{1}{\sqrt{2\pi}} e^{\pm i\phi}$	$\frac{\sqrt{3}}{2}\sin\theta$	$\frac{1}{2\sqrt{6}\, a_0{}^{3/2}}\frac{r}{a_0} e^{-r/2a_0}$	$\frac{1}{8\sqrt{\pi}\, a_0{}^{3/2}}\frac{r}{a_0} e^{-r/2a_0}\sin\theta\, e^{\pm i\phi}$
3	0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2}}$	$\frac{2}{81\sqrt{3}\, a_0{}^{3/2}}\left(27-18\frac{r}{a_0}+2\frac{r^2}{a_0{}^2}\right) e^{-r/3a_0}$	$\frac{1}{81\sqrt{3\pi}\, a_0{}^{3/2}}\left(27-18\frac{r}{a_0}+2\frac{r^2}{a_0{}^2}\right) e^{-r/3a_0}$
3	1	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{\sqrt{6}}{2}\cos\theta$	$\frac{4}{81\sqrt{6}\, a_0{}^{3/2}}\left(6-\frac{r}{a_0}\right)\frac{r}{a_0} e^{-r/3a_0}$	$\frac{\sqrt{2}}{81\sqrt{\pi}\, a_0{}^{3/2}}\left(6-\frac{r}{a_0}\right)\frac{r}{a_0} e^{-r/3a_0}\cos\theta$
3	1	±1	$\frac{1}{\sqrt{2\pi}} e^{\pm i\phi}$	$\frac{\sqrt{3}}{2}\sin\theta$	$\frac{4}{81\sqrt{6}\, a_0{}^{3/2}}\left(6-\frac{r}{a_0}\right)\frac{r}{a_0} e^{-r/3a_0}$	$\frac{1}{81\sqrt{\pi}\, a_0{}^{3/2}}\left(6-\frac{r}{a_0}\right)\frac{r}{a_0} e^{-r/3a_0}\sin\theta\, e^{\pm i\phi}$
3	2	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{\sqrt{10}}{4}(3\cos^2\theta-1)$	$\frac{4}{81\sqrt{30}\, a_0{}^{3/2}}\frac{r^2}{a_0{}^2} e^{-r/3a_0}$	$\frac{1}{81\sqrt{6\pi}\, a_0{}^{3/2}}\frac{r^2}{a_0{}^2} e^{-r/3a_0}(3\cos^2\theta-1)$
3	2	±1	$\frac{1}{\sqrt{2\pi}} e^{\pm i\phi}$	$\frac{\sqrt{15}}{2}\sin\theta\cos\theta$	$\frac{4}{81\sqrt{30}\, a_0{}^{3/2}}\frac{r^2}{a_0{}^2} e^{-r/3a_0}$	$\frac{1}{81\sqrt{\pi}\, a_0{}^{3/2}}\frac{r^2}{a_0{}^2} e^{-r/3a_0}\sin\theta\cos\theta\, e^{\pm i\phi}$
3	2	±2	$\frac{1}{\sqrt{2\pi}} e^{\pm 2i\phi}$	$\frac{\sqrt{15}}{4}\sin^2\theta$	$\frac{4}{81\sqrt{30}\, a_0{}^{3/2}}\frac{r^2}{a_0{}^2} e^{-r/3a_0}$	$\frac{1}{162\sqrt{\pi}\, a_0{}^{3/2}}\frac{r^2}{a_0{}^2} e^{-r/3a_0}\sin^2\theta\, e^{\pm 2i\phi}$

Hydrogen atom Solutions

- Principal quantum number, $n = 1, 2, 3, \dots$
- Orbital quantum number, $l = 0, 1, 2, \dots (n-1)$ where l =1*(s)*, =2*(p)*, =3*(d)*, =4*(f)*, *etc.*
- Magnetic quantum number, $m_l = 0, \pm 1, \pm 2, \dots \pm l$

Wave function = $\psi(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi) = R_{n,l}\Theta_{l,m_l}\Phi_{m_l}$

where $\Theta_{l,m_l}\Phi_{m_l} = \mathbf{Y}_l^{m_l}$ (Spherical harmonics)

n	l	m_l	$\Phi(\phi)$	$\Theta(\theta)$	$R(r)$	$\psi(r,\theta,\phi)$
1	0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2}}$	$\frac{2}{a_0{}^{3/2}} e^{-r/a_0}$	$\frac{1}{\sqrt{\pi}\, a_0{}^{3/2}} e^{-r/a_0}$
2	0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2\sqrt{2}\, a_0{}^{3/2}} \left(2 - \frac{r}{a_0}\right) e^{-r/2a_0}$	$\frac{1}{4\sqrt{2\pi}\, a_0{}^{3/2}} \left(2 - \frac{r}{a_0}\right) e^{-r/2a_0}$
2	1	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{\sqrt{6}}{2} \cos\theta$	$\frac{1}{2\sqrt{6}\, a_0{}^{3/2}} \frac{r}{a_0} e^{-r/2a_0}$	$\frac{1}{4\sqrt{2\pi}\, a_0{}^{3/2}} \frac{r}{a_0} e^{-r/2a_0} \cos\theta$
2	1	± 1	$\frac{1}{\sqrt{2\pi}} e^{\pm i\phi}$	$\frac{\sqrt{3}}{2} \sin\theta$	$\frac{1}{2\sqrt{6}\, a_0{}^{3/2}} \frac{r}{a_0} e^{-r/2a_0}$	$\frac{1}{8\sqrt{\pi}\, a_0{}^{3/2}} \frac{r}{a_0} e^{-r/2a_0} \sin\theta\, e^{\pm i\phi}$
3	0	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{1}{\sqrt{2}}$	$\frac{2}{81\sqrt{3}\, a_0{}^{3/2}} \left(27 - 18\frac{r}{a_0} + 2\frac{r^2}{a_0{}^2}\right) e^{-r/3a_0}$	$\frac{1}{81\sqrt{3\pi}\, a_0{}^{3/2}} \left(27 - 18\frac{r}{a_0} + 2\frac{r^2}{a_0{}^2}\right) e^{-r/3a_0}$
3	1	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{\sqrt{6}}{2} \cos\theta$	$\frac{4}{81\sqrt{6}\, a_0{}^{3/2}} \left(6 - \frac{r}{a_0}\right) \frac{r}{a_0} e^{-r/3a_0}$	$\frac{\sqrt{2}}{81\sqrt{\pi}\, a_0{}^{3/2}} \left(6 - \frac{r}{a_0}\right) \frac{r}{a_0} e^{-r/3a_0} \cos\theta$
3	1	± 1	$\frac{1}{\sqrt{2\pi}} e^{\pm i\phi}$	$\frac{\sqrt{3}}{2} \sin\theta$	$\frac{4}{81\sqrt{6}\, a_0{}^{3/2}} \left(6 - \frac{r}{a_0}\right) \frac{r}{a_0} e^{-r/3a_0}$	$\frac{1}{81\sqrt{\pi}\, a_0{}^{3/2}} \left(6 - \frac{r}{a_0}\right) \frac{r}{a_0} e^{-r/3a_0} \sin\theta\, e^{\pm i\phi}$
3	2	0	$\frac{1}{\sqrt{2\pi}}$	$\frac{\sqrt{10}}{4} (3\cos^2\theta - 1)$	$\frac{4}{81\sqrt{30}\, a_0{}^{3/2}} \frac{r^2}{a_0{}^2} e^{-r/3a_0}$	$\frac{1}{81\sqrt{6\pi}\, a_0{}^{3/2}} \frac{r^2}{a_0{}^2} e^{-r/3a_0} (3\cos^2\theta - 1)$
3	2	± 1	$\frac{1}{\sqrt{2\pi}} e^{\pm i\phi}$	$\frac{\sqrt{15}}{2} \sin\theta\cos\theta$	$\frac{4}{81\sqrt{30}\, a_0{}^{3/2}} \frac{r^2}{a_0{}^2} e^{-r/3a_0}$	$\frac{1}{81\sqrt{\pi}\, a_0{}^{3/2}} \frac{r^2}{a_0{}^2} e^{-r/3a_0} \sin\theta\cos\theta\, e^{\pm i\phi}$
3	2	± 2	$\frac{1}{\sqrt{2\pi}} e^{\pm 2i\phi}$	$\frac{\sqrt{15}}{4} \sin^2\theta$	$\frac{4}{81\sqrt{30}\, a_0{}^{3/2}} \frac{r^2}{a_0{}^2} e^{-r/3a_0}$	$\frac{1}{162\sqrt{\pi}\, a_0{}^{3/2}} \frac{r^2}{a_0{}^2} e^{-r/3a_0} \sin^2\theta\, e^{\pm 2i\phi}$

Schrodinger Equation: Hydrogen Atom

$$\frac{-\hbar^2}{2m}\nabla^2\psi(\vec{x}) + U(\vec{x})\psi(\vec{x}) = E\,\psi(\vec{x}) \qquad U(r) = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}$$

$$\left[\frac{-\hbar^2}{2m}\nabla^2 + U(\vec{x})\right]\psi(\vec{x}) = E\,\psi(\vec{x})$$

Hamiltonian (H) = T (kinetic) + U (Potential)

Kinetic energy w.r.t. r + Kinetic energy w.r.t. rotation

$$\begin{cases} -\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - \frac{2mr^2}{\hbar^2}\left(E - U(r)\right) = C = -l(l+1) \\ \frac{1}{\Theta}\csc\theta\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\Phi}\csc^2\theta\,\frac{\partial^2\Phi}{\partial\phi^2} = C = -l(l+1) \end{cases}$$

$$\frac{\partial^2\Phi}{\partial\phi^2} = -m_l^2\,\Phi$$

Schrodinger Equation: Hydrogen Atom

$$\frac{-\hbar^2}{2m}\nabla^2\psi(\vec{x}) + U(\vec{x})\psi(\vec{x}) = E\,\psi(\vec{x}) \qquad U(r) = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}$$

$$\left[\frac{-\hbar^2}{2m}\nabla^2 + U(\vec{x})\right]\psi(\vec{x}) = E\,\psi(\vec{x})$$

$$\mathbf{H}\psi_{n,l,m_l} = E_n\psi_{n,l,m_l}$$

Hamiltonian (H) = T (kinetic) + U (Potential)

Kinetic energy w.r.t. r + Kinetic energy w.r.t. rotation

$$\begin{cases} -\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - \frac{2mr^2}{\hbar^2}\left(E - U(r)\right) = C = -l(l+1) \\ \frac{1}{\Theta}\csc\theta\,\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\Phi}\csc^2\theta\frac{\partial^2\Phi}{\partial\phi^2} = C = -l(l+1) \end{cases}$$

$$\mathbf{L}^2\psi_{n,l,m_l} = l(l+1)\hbar^2\psi_{n,l,m_l}$$

$$\frac{\partial^2\Phi}{\partial\phi^2} = -m_l^2\Phi$$

$$\mathbf{L_z}\psi_{n,l,m_l} = m_l\hbar\psi_{n,l,m_l}$$

The angular momentum **L** of a particle at the position **r** whose linear momentum is **p** is defined by the vector formula

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

(a) From the above expression, explain how three Cartesian components of **L** can be written as:

$$\begin{cases} L_x = yp_z - zp_y \\ L_y = zp_x - xp_z \\ L_z = xp_y - yp_x \end{cases}$$

(b) Using the momentum operators such as $p_x = \frac{\hbar}{i}\frac{\partial}{\partial x}$, etc. express three Cartesian angular momentum operators L_x, L_y, L_z.

(c) Prove that the three Cartesian angular momentum operators in spherical polar coordinates can be written as:

$$L_x = \frac{\hbar}{i}\left(-\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)$$

$$L_y = \frac{\hbar}{i}\left(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)$$

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$$

(d) $\mathbf{L}^2$ operator is defined by:

$$\mathbf{L}^2 = \mathbf{L_x L_x} + \mathbf{L_y L_y} + \mathbf{L_z L_z}$$

Prove that

$$\mathbf{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

(e) Using $L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$ **and** : $\Phi(\phi) = A\,e^{im_l\phi}$ **prove that**

$$L_z\psi = m_l\hbar\psi$$

(f) Using $\mathbf{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$

$\Phi(\phi) = A\,e^{im_l\phi}$

$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) + [l(l+1)\sin^2\theta - m_l^2]\Theta = 0$

Prove that

$$L^2\,\psi = l(l+1)\hbar^2\psi$$

Three quantum numbers

- Principal quantum number: *n*

$$E_n = -\frac{me^4}{32\pi^2\varepsilon_0^2\hbar^2}\left(\frac{1}{n^2}\right) = -\left(\frac{e^2}{8\pi\varepsilon_0}\right)\left(\frac{me^2}{4\pi\varepsilon_0\hbar^2}\right)\left(\frac{1}{n^2}\right) =$$

$$-\left(\frac{e^2}{8\pi\varepsilon_0 a_0}\right)\left(\frac{1}{n^2}\right) = -13.6\ eV\left(\frac{1}{n^2}\right)$$

$$L^2 = l(l+1)\hbar^2$$

$$L_z = m_l\hbar$$

- Orbital quantum number: $l = 0, 1, 2, \ldots (n\text{-}1)$
- Magnetic quantum number: $m_l = 0, \pm 1, \pm 2, \ldots \pm l$

Degeneracies

$$\psi(r,\theta,\phi) = \psi_{n,l,m_l} = R(r)\Theta(\theta)\Phi(\phi) = R_{n,l}\Theta_{l,m_l}\Phi_{m_l} = R_{n,l}\mathbf{Y}_l^{m_l}$$

where $\Theta_{l,m_l}\Phi_{m_l} = \mathbf{Y}_l^{m_l}$ (Spherical harmonics)

$$\mathbf{L}^2\psi_{n,l,m_l} = \boldsymbol{l}(\boldsymbol{l}+\mathbf{1})\hbar^2\psi_{n,l,m_l}$$

$$\mathbf{L_z}\psi_{n,l,m_l} = \boldsymbol{m_l}\hbar\psi_{n,l,m_l}$$

n	l	m_l	E_n(eV)	$\lvert L\rvert$	L_z	ψ_{n,l,m_l} =	$R_{n,l}\mathbf{Y}_l^{m_l}$	degeneracies	Orbital name
1	0	0	-13.6	0	0	ψ_{100}	$R_{10}\mathbf{Y}_0^0$	Non-degenerate	1s

-

Degeneracies

$$\psi(r,\theta,\phi) = \psi_{n,l,m_l} = R(r)\Theta(\theta)\Phi(\phi) = R_{n,l}\Theta_{l,m_l}\Phi_{m_l} = R_{n,l}\mathbf{Y}_l^{m_l}$$

where $\Theta_{l,m_l}\Phi_{m_l} = \mathbf{Y}_l^{m_l}$ (Spherical harmonics)

$$\mathbf{L}^2\psi_{n,l,m_l} = \boldsymbol{l}(\boldsymbol{l}+\mathbf{1})\hbar^2\psi_{n,l,m_l}$$

$$\mathbf{L}_z\psi_{n,l,m_l} = \boldsymbol{m}_l\hbar\psi_{n,l,m_l}$$

n	l	m_l	E_n(eV)	$\lvert L\rvert$	L_z	ψ_{n,l,m_l} =	$R_{n,l}\mathbf{Y}_l^{m_l}$	degeneracies	Orbital name
1	0	0	-13.6	0	0	ψ_{100}	$R_{10}\mathbf{Y}_0^0$	Non-degenerate	1s
2	0	0							
	1	-1							
		0							
		1							

Degeneracies

$$\psi(r,\theta,\phi) = \psi_{n,l,m_l} = R(r)\Theta(\theta)\Phi(\phi) = R_{n,l}\Theta_{l,m_l}\Phi_{m_l} = R_{n,l}\mathbf{Y}_l^{m_l}$$

where $\Theta_{l,m_l}\Phi_{m_l} = \mathbf{Y}_l^{m_l}$ (Spherical harmonics)

$$\mathbf{L}^2\psi_{n,l,m_l} = l(l+1)\hbar^2\psi_{n,l,m_l}$$

$$\mathbf{L_z}\psi_{n,l,m_l} = m_l\hbar\psi_{n,l,m_l}$$

n	l	m_l	E_n(eV)	$\lvert L\rvert$	L_z	ψ_{n,l,m_l} =	$R_{n,l}\mathbf{Y}_l^{m_l}$	degeneracies	Orbital name
1	0	0	-13.6	0	0	ψ_{100}	$R_{10}\mathbf{Y}_0^0$	Non-degenerate	1s
2	0	0	-3.40	0	0				
	1	-1		$\sqrt{2}\hbar$	$-\hbar$				
		0			0				
		1			$+\hbar$				

Degeneracies

$$\psi(r,\theta,\phi) = \psi_{n,l,m_l} = R(r)\Theta(\theta)\Phi(\phi) = R_{n,l}\Theta_{l,m_l}\Phi_{m_l} = R_{n,l}\mathbf{Y}_l^{m_l}$$

where $\Theta_{l,m_l}\Phi_{m_l} = \mathbf{Y}_l^{m_l}$ (Spherical harmonics)

$$\mathbf{L}^2\psi_{n,l,m_l} = l(l+1)\hbar^2\psi_{n,l,m_l}$$

$$\mathbf{L_z}\psi_{n,l,m_l} = m_l\hbar\psi_{n,l,m_l}$$

n	l	m_l	E_n(eV)	$\lvert L\rvert$	L_z	ψ_{n,l,m_l} =	$R_{n,l}\mathbf{Y}_l^{m_l}$	degeneracies	Orbital name
1	0	0	-13.6	0	0	ψ_{100}	$R_{10}\mathbf{Y}_0^0$	Non-degenerate	1s
2	0	0	-3.40	0	0	ψ_{200}	$R_{20}\mathbf{Y}_0^0$		
	1	-1		$\sqrt{2}\hbar$	$-\hbar$	ψ_{21-1}	$R_{21}\mathbf{Y}_1^{-1}$		
		0			0	ψ_{210}	$R_{21}\mathbf{Y}_1^0$		
		1			$+\hbar$	ψ_{211}	$R_{21}\mathbf{Y}_1^{+1}$		

Degeneracies

$$\psi(r,\theta,\phi) = \psi_{n,l,m_l} = R(r)\Theta(\theta)\Phi(\phi) = R_{n,l}\Theta_{l,m_l}\Phi_{m_l} = R_{n,l}\mathbf{Y}_l^{m_l}$$

where $\Theta_{l,m_l}\Phi_{m_l} = \mathbf{Y}_l^{m_l}$ (Spherical harmonics)

$$\mathbf{L}^2\psi_{n,l,m_l} = \boldsymbol{l(l+1)}\hbar^2\psi_{n,l,m_l}$$

$$\mathbf{L_z}\psi_{n,l,m_l} = \boldsymbol{m_l}\hbar\psi_{n,l,m_l}$$

n	l	m_l	E_n(eV)	$\lvert L\rvert$	L_z	ψ_{n,l,m_l} =	$R_{n,l}\mathbf{Y}_l^{m_l}$	degeneracies	Orbital name
1	0	0	-13.6	0	0	ψ_{100}	$R_{10}\mathbf{Y}_0^0$	Non-degenerate	1s
2	0	0	-3.40	0	0	ψ_{200}	$R_{20}\mathbf{Y}_0^0$	4 (=2^2)	2s
	1	-1		$\sqrt{2}\hbar$	$-\hbar$	ψ_{21-1}	$R_{21}\mathbf{Y}_1^{-1}$		2p
		0			0	ψ_{210}	$R_{21}\mathbf{Y}_1^0$		
		1			$+\hbar$	ψ_{211}	$R_{21}\mathbf{Y}_1^{+1}$		

n	l	m_l	E_n(eV)	$\lvert L\rvert$	L_z	ψ_{n,l,m_l} =	$R_{n,l}\mathbf{Y}_l^{m_l}$	degeneracies	Orbital name
1	0	0	-13.6	0	0	ψ_{100}	$R_{10}\mathbf{Y}_0^0$	Non-degenerate	1s
2	0	0	-3.40	0	0	ψ_{200}	$R_{20}\mathbf{Y}_0^0$	4 (=2^2)	2s
	1	-1		$\sqrt{2}\hbar$	$-\hbar$	ψ_{21-1}	$R_{21}\mathbf{Y}_1^{-1}$		2p
		0			0	ψ_{210}	$R_{21}\mathbf{Y}_1^0$		
		1			$+\hbar$	ψ_{211}	$R_{21}\mathbf{Y}_1^{+1}$		
3	0	0	-1.51						
	1	-1							
		0							
		1							
	2	-2							
		--1							
		0							
		1							
		2							

n	l	m_l	E_n(eV)	$\|L\|$	L_z	ψ_{n,l,m_l} =	$R_{n,l}\mathbf{Y}_l^{m_l}$	degeneracies	Orbital name
1	0	0	-13.6	0	0	ψ_{100}	$R_{10}\mathbf{Y}_0^0$	Non-degenerate	1s
2	0	0	-3.40	0	0	ψ_{200}	$R_{20}\mathbf{Y}_0^0$	4 (=2^2)	2s
	1	-1		$\sqrt{2}\hbar$	$-\hbar$	ψ_{21-1}	$R_{21}\mathbf{Y}_1^{-1}$		2p
		0			0	ψ_{210}	$R_{21}\mathbf{Y}_1^0$		
		1			$+\hbar$	ψ_{211}	$R_{21}\mathbf{Y}_1^{+1}$		
3	0	0	-1.51	0	0	ψ_{300}	$R_{30}\mathbf{Y}_0^0$	9 (=3^2)	3s
	1	-1		$\sqrt{2}\hbar$	$-\hbar$	ψ_{31-1}	$R_{31}\mathbf{Y}_1^{-1}$		3p
		0			0	ψ_{310}	$R_{31}\mathbf{Y}_1^0$		
		1			$+\hbar$	ψ_{311}	$R_{31}\mathbf{Y}_1^1$		
	2	-2		$\sqrt{6}\hbar$	$-2\hbar$	ψ_{32-2}	$R_{32}\mathbf{Y}_2^{-2}$		3d
		--1			$-\hbar$	ψ_{32-1}	$R_{32}\mathbf{Y}_2^{-1}$		
		0			0	ψ_{320}	$R_{32}\mathbf{Y}_2^0$		
		1			$+\hbar$	ψ_{321}	$R_{32}\mathbf{Y}_2^1$		
		2			$+2\hbar$	ψ_{322}	$R_{32}\mathbf{Y}_2^2$		