Lecture 9 Topics

- Three occupation number plots
- Classical limits
- Fermi Gas
  - Fermi energy
  - Cold metal
  - Contact potential
- Boson Gas
  - Superfluid: He_4

System requirement

- Conservation of particles

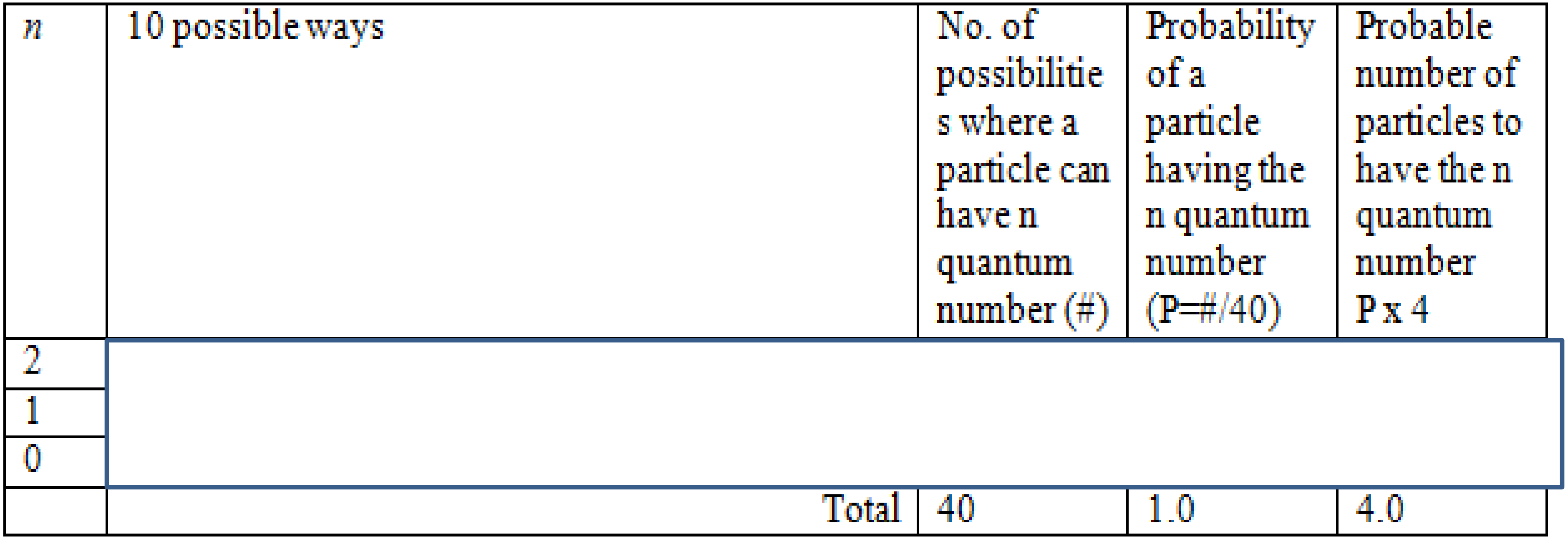
$$\sum \mathcal{N}(E_i) = \mathcal{N}(E_1) + \mathcal{N}(E_2) + \cdots + \mathcal{N}(E_k) = N$$

- Conservation of energy

$$\sum \mathcal{N}(E_i)E_i = \mathcal{N}(E_1)E_1 + \mathcal{N}(E_2)E_2 + \cdots + \mathcal{N}(E_k)E_k = E$$

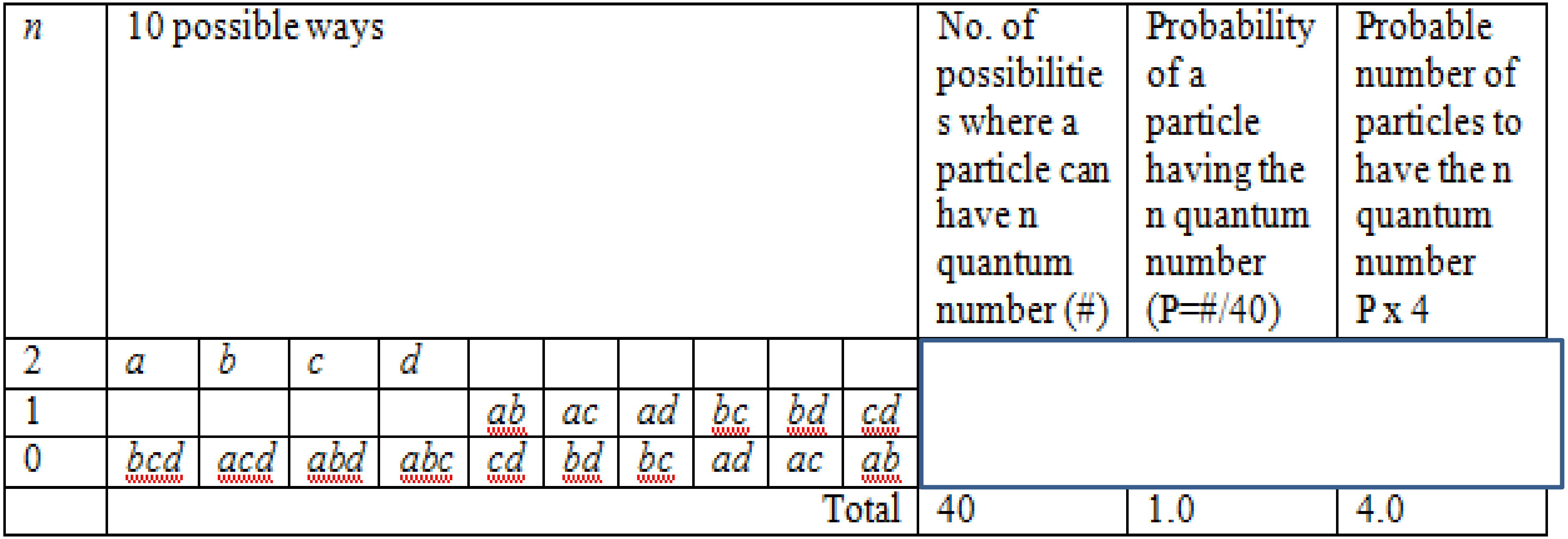
- Example: 4 (a, b, c, d) particles with total energy of $2\hbar\omega_0$.
  - Consider $E_i = n_i\hbar\omega_0$

Maxwell-Boltzman



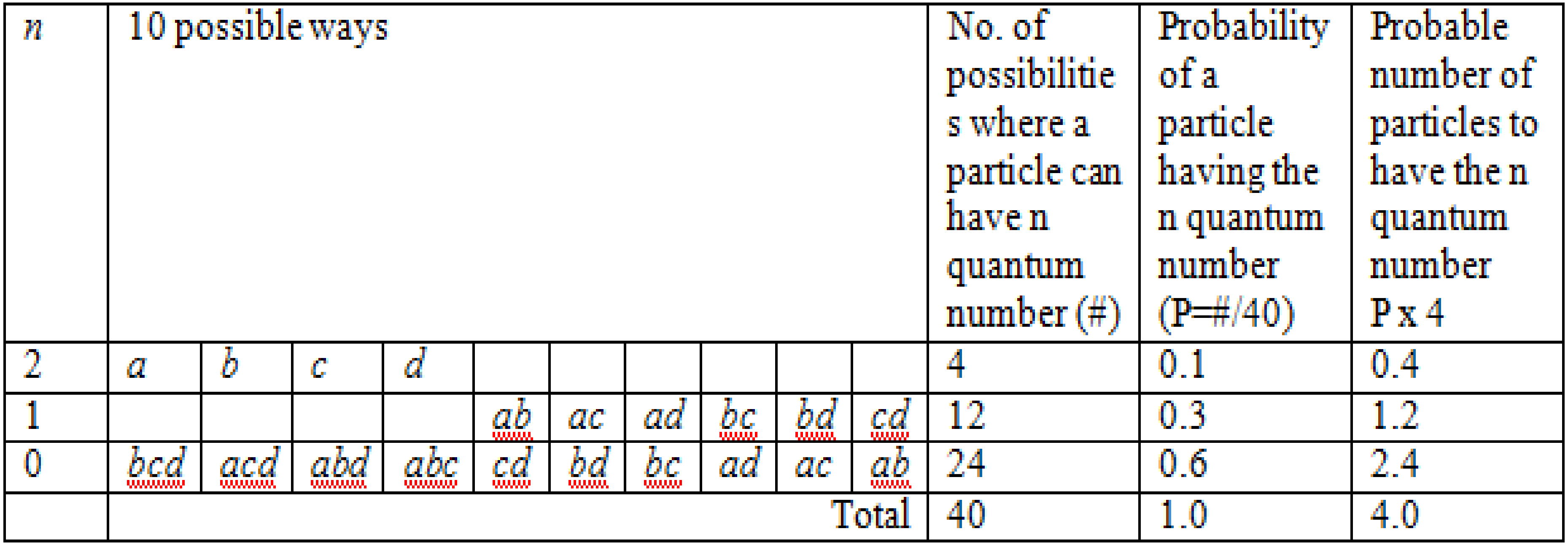
n	10 possible ways	No. of possibilities where a particle can have n quantum number (#)	Probability of a particle having the n quantum number (P=#/40)	Probable number of particles to have the n quantum number P x 4
2				
1				
0				
	Total	40	1.0	4.0

Maxwell-Boltzman



n	10 possible ways										No. of possibilities where a particle can have n quantum number (#)	Probability of a particle having the n quantum number (P=#/40)	Probable number of particles to have the n quantum number P x 4
2	*a*	*b*	*c*	*d*									
1					*ab*	*ac*	*ad*	*bc*	*bd*	*cd*			
0	*bcd*	*acd*	*abd*	*abc*	*cd*	*bd*	*bc*	*ad*	*ac*	*ab*			
										Total	40	1.0	4.0

Maxwell-Boltzman

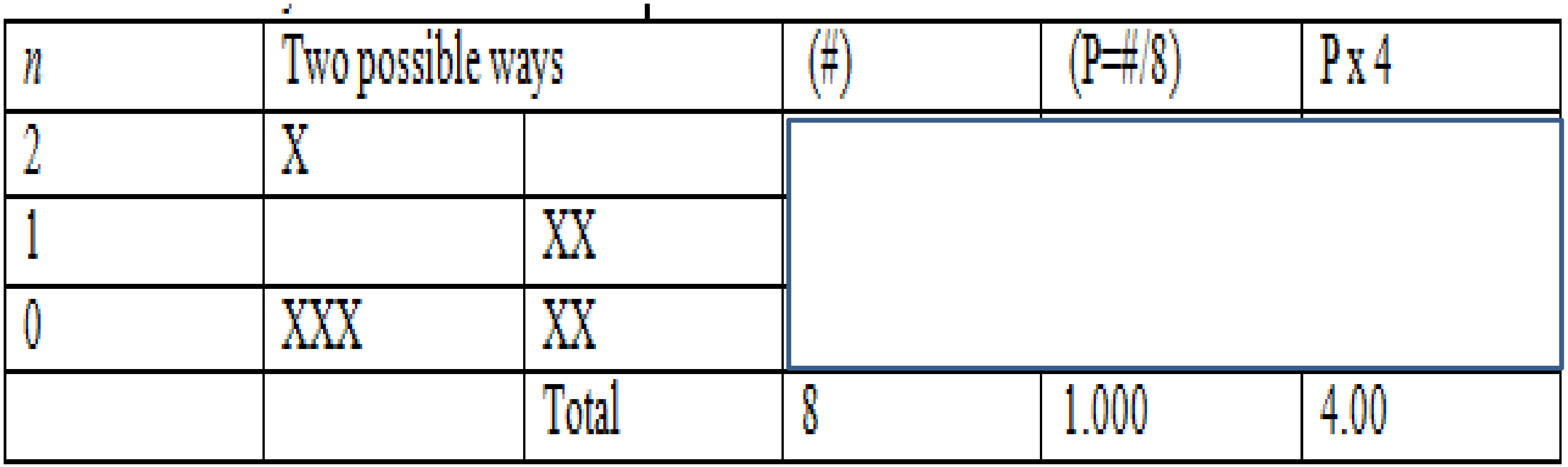


n	10 possible ways										No. of possibilities where a particle can have n quantum number (#)	Probability of a particle having the n quantum number (P=#/40)	Probable number of particles to have the n quantum number P x 4
2	*a*	*b*	*c*	*d*							4	0.1	0.4
1					*ab*	*ac*	*ad*	*bc*	*bd*	*cd*	12	0.3	1.2
0	*bcd*	*acd*	*abd*	*abc*	*cd*	*bd*	*bc*	*ad*	*ac*	*ab*	24	0.6	2.4
	Total										40	1.0	4.0

Bose-Einstein

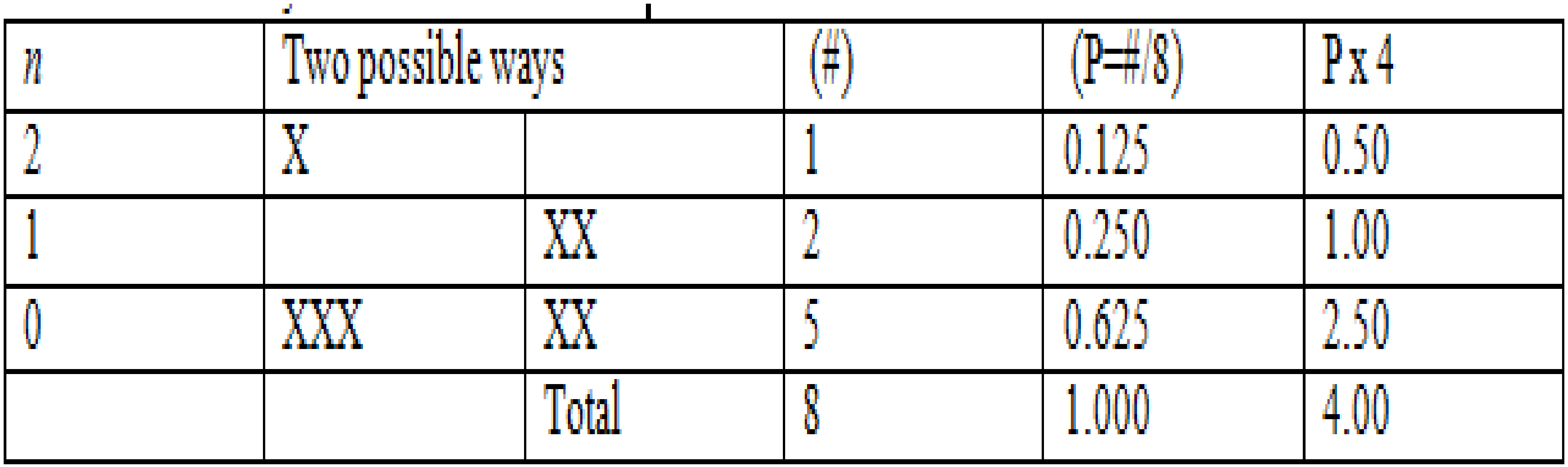
n	Two possible ways		(#)	(P=#/8)	P x 4
2					
1					
0					
		Total	8	1.000	4.00

Bose-Einstein



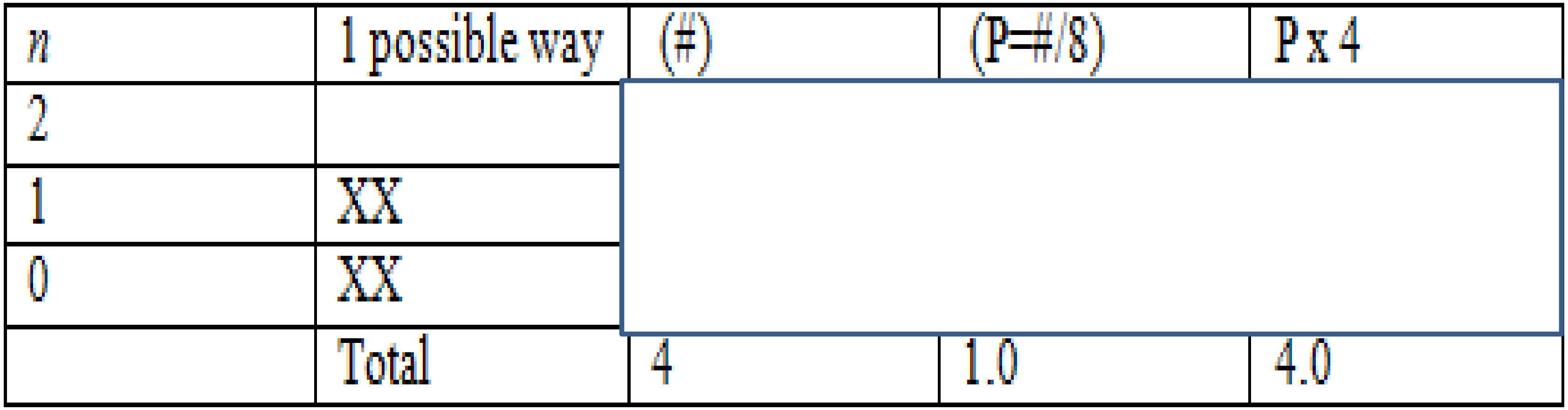
n	Two possible ways		(#)	(P=#/8)	P x 4
2	X				
1		XX			
0	XXX	XX			
		Total	8	1.000	4.00

Bose-Einstein



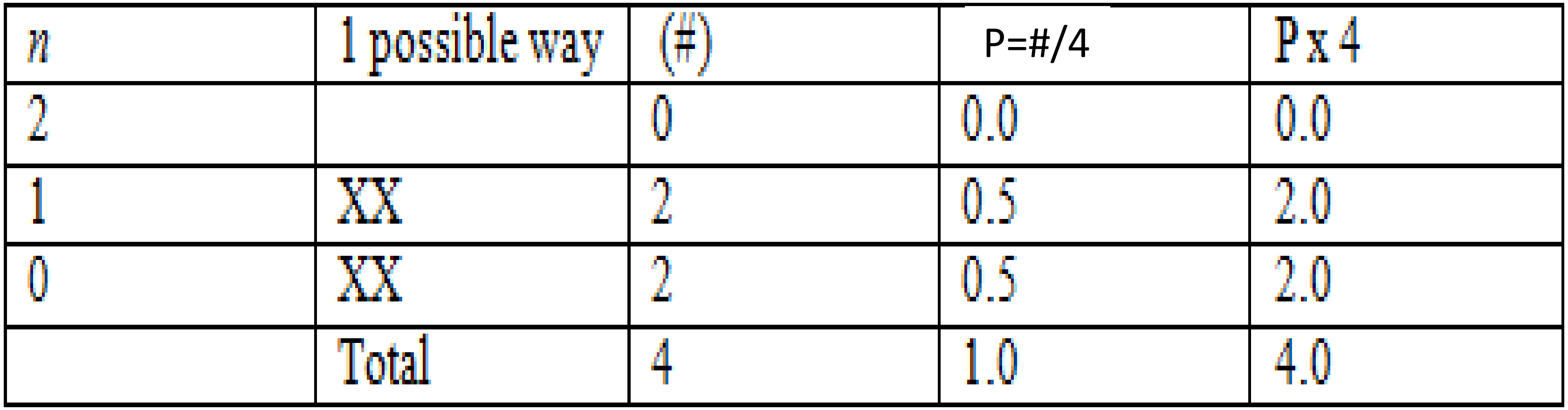
n	Two possible ways		(#)	(P=#/8)	P x 4
2	X		1	0.125	0.50
1		XX	2	0.250	1.00
0	XXX	XX	5	0.625	2.50
		Total	8	1.000	4.00

Fermi-Dirac



n	1 possible way	(#)	(P=#/8)	P x 4
2				
1	XX			
0	XX			
	Total	4	1.0	4.0

Fermi-Dirac

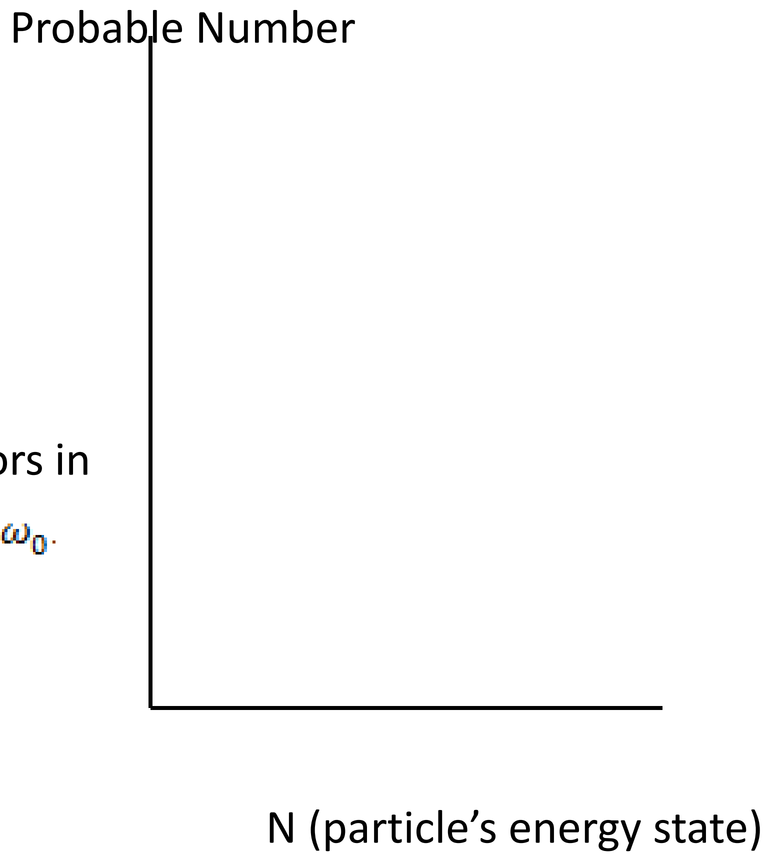


n	1 possible way	(#)	P=#/4	P x 4
2		0	0.0	0.0
1	XX	2	0.5	2.0
0	XX	2	0.5	2.0
	Total	4	1.0	4.0

Probable Number Plotting

n	Maxwell-Boltzmann	Bose-Einstein	Fermi-Dirac
2	.4	0.5	0.0
1	1.2	1.0	2.0
0	2.4	2.5	2.0

When there are four harmonic oscillators in a system with the system's energy is $2\hbar\omega_0$.



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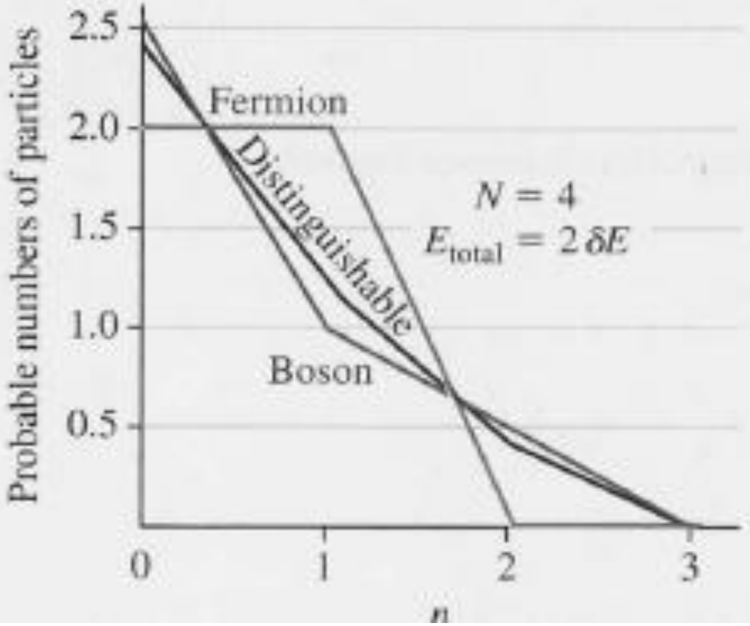
When there are four harmonic oscillators in a system with the system's energy is $2\hbar\omega_0$.

Figure 9.9 The probable number of particles at the allowed energies depends on whether the particles are bosons, fermions, or distinguishable.



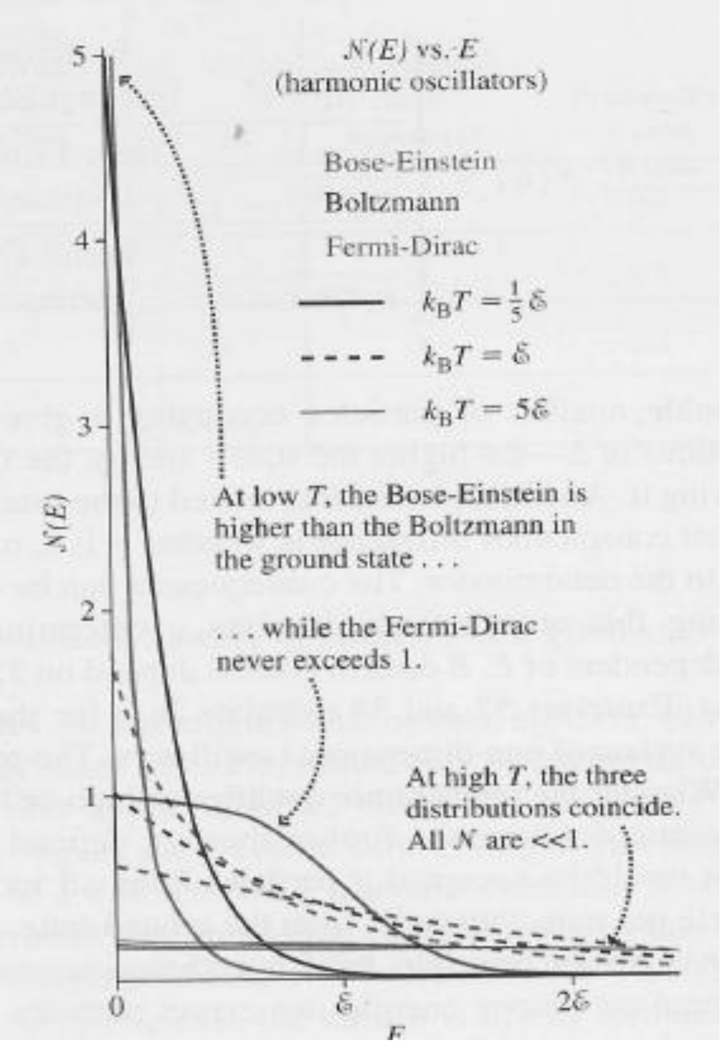
Occupation number distribution comparison

As E increases

$k_BT >> E$ → High temp relative to system E
Thin gray lines

$k_BT \approx E$
Dashed lines

$k_BT << E$ → Low temp relative to system E
Solid lines



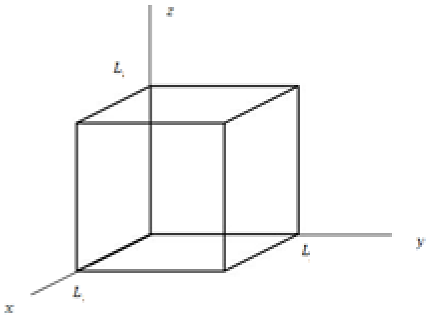
At low T, the Bose-Einstein is higher than the Boltzmann in the ground state . . .

. . . while the Fermi-Dirac never exceeds 1.

At high T, the three distributions coincide. All N are <<1.

When to use quantum gas treatment?

- A system of N particles in a 3-D box

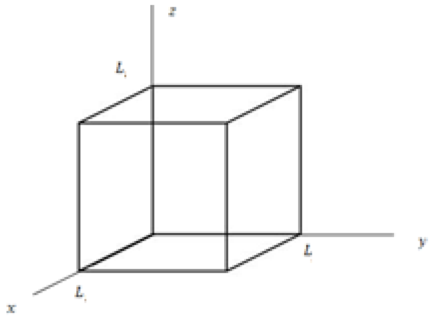


The 3D Box

$$U(\vec{x}) = \begin{cases} 0 & 0 < x, y, z < L, \\ \infty & \text{otherwise} \end{cases}$$

When to use quantum gas treatment?

- A system of N particles in a 3-D box



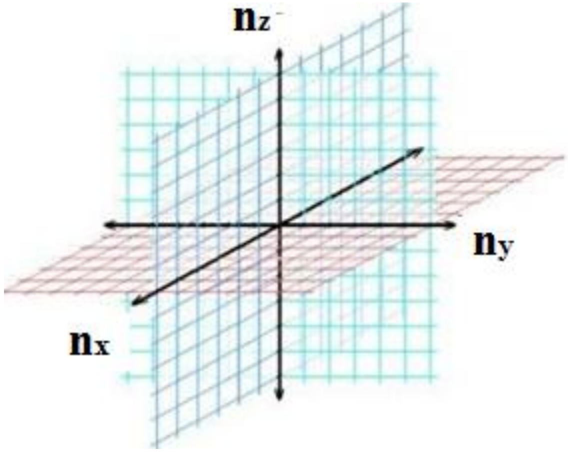
The 3D Box

$$U(\vec{x}) = \begin{cases} 0 & 0 < x, y, z < L, \\ \infty & \quad\quad otherwise \end{cases}$$

$$\psi\,(x, y, z) = A\,\sin\frac{n_x \pi x}{L}\sin\frac{n_y \pi y}{L}\sin\frac{n_z \pi z}{L}$$

$$E_{(n_x, n_y, n_z)} = (n_x^2 + n_y^2 + n_z^2)\,\frac{\pi^2\hbar^2}{2mL^2}$$

Quantum Number Space



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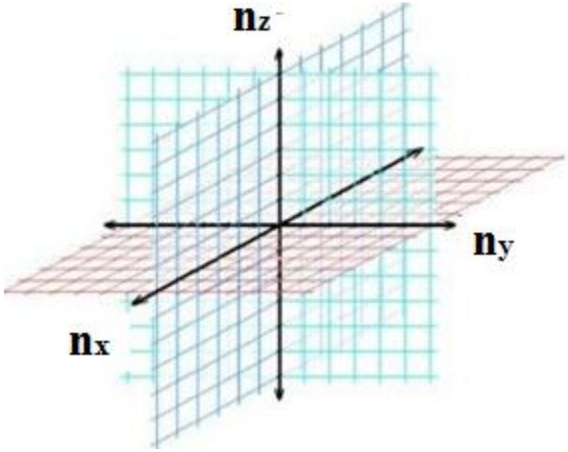

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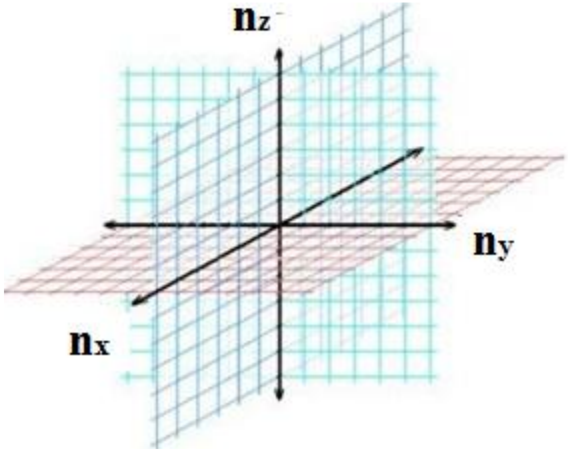
$$n^2 = n_x^2 + n_y^2 + n_z^2$$

$$E = \frac{\pi^2\hbar^2 n^2}{2mL^2} \qquad n = \sqrt{\frac{2mL^2}{\pi^2\hbar^2}}\sqrt{E}$$

Quantum Number Space



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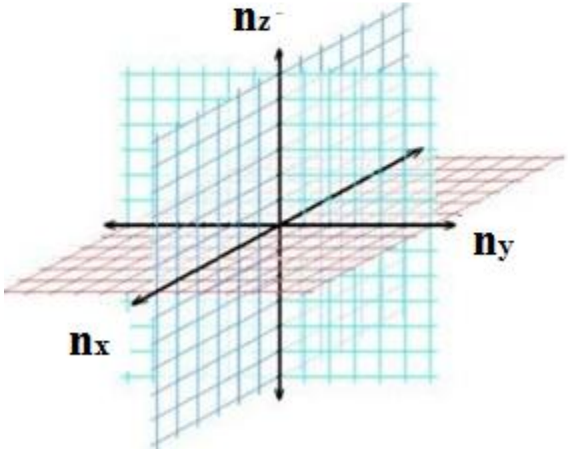
Quantum Number Space



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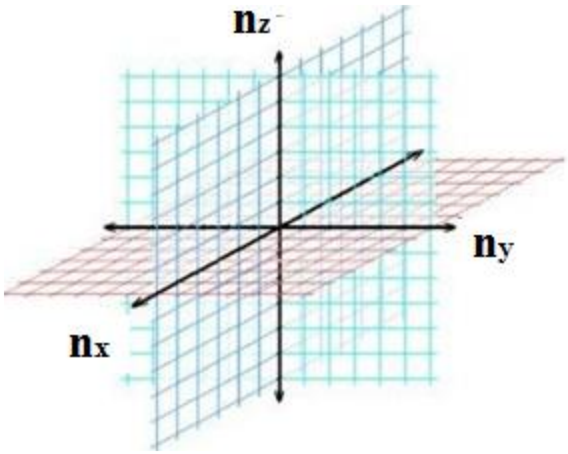
$$D(E) = \frac{No.of\ states\ at\ E + dE - No.of\ states\ at\ E}{dE} =$$ [redacted] $$\frac{dn}{dE}$$

Quantum Number Space



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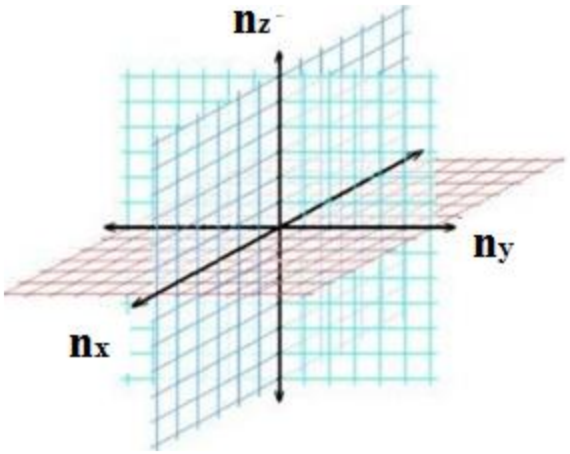
Quantum Number Space



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Particles that obey the Boltzman distribution

$$\mathcal{N}(E) = \frac{1}{Be^{E/k_BT}}$$

Particles that obey the Bose-Einstein distribution

$$\mathcal{N}(E) = \frac{1}{Be^{E/k_BT} - 1}$$

Particles that obey the Fermi-Dirac distribution

$$\mathcal{N}(E) = \frac{1}{Be^{E/k_BT} + 1}$$

$$D(E) = (2s+1)\frac{m^{3/2}V}{\pi^3\hbar^3\sqrt{2}}\sqrt{E}$$

$$N = \int_0^\infty \mathcal{N}(E)D(E)dE =$$

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$$\bar{E} = \frac{\int_0^\infty E\mathcal{N}(E)D(E)dE}{\int_0^\infty \mathcal{N}(E)D(E)dE} =$$

[redacted]

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Classical Limit: $$\frac{\hbar^{3}}{(mk_BT)^{3/2}}\left(\frac{N}{V}\right)^{1} \ll 1$$

$$D(E) = (2s+1)\frac{m^{3/2}V}{\pi^3\hbar^3\sqrt{2}}\sqrt{E}$$

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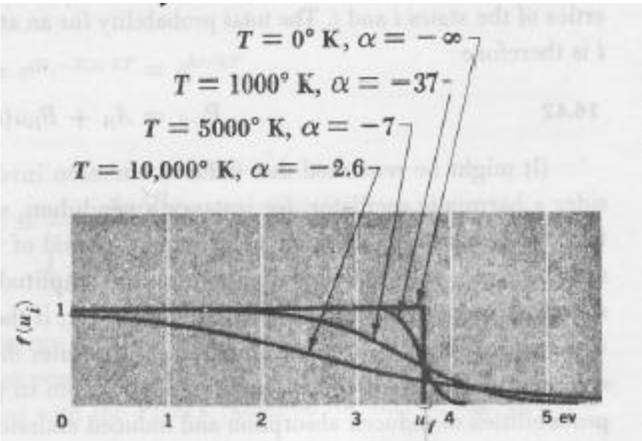
For the ideal gas, this term is In the order of $\cong 9 \text{ x } 10^{-9}$

Classical Limit: $\frac{\hbar^3}{(mk_BT)^{3/2}}(\frac{N}{V})^1 \ll 1$ OR $(\frac{\lambda}{d})^3 \ll 1$

Fermi Energy



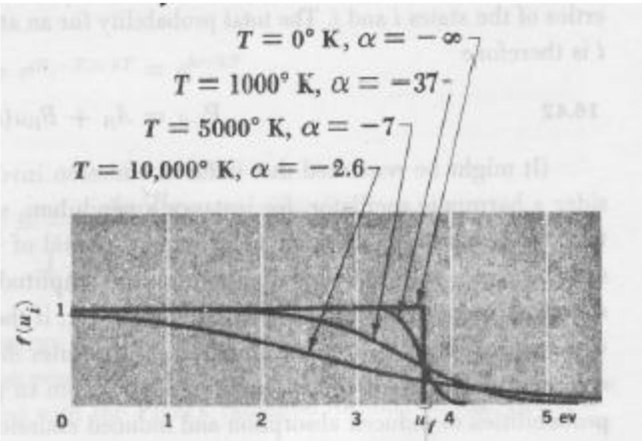
At T=0, electrons fill upto the E_F Level
There are N electrons.

$$N = \int \mathcal{N}(E) D(E) dE$$

$$\mathcal{N}(E) = \frac{1}{e^{(E-E_F)/k_B T} + 1}$$

$$D(E) = (2s+1)\frac{m^{3/2}V}{\pi^3\hbar^3\sqrt{2}}\sqrt{E}$$

Fermi Energy



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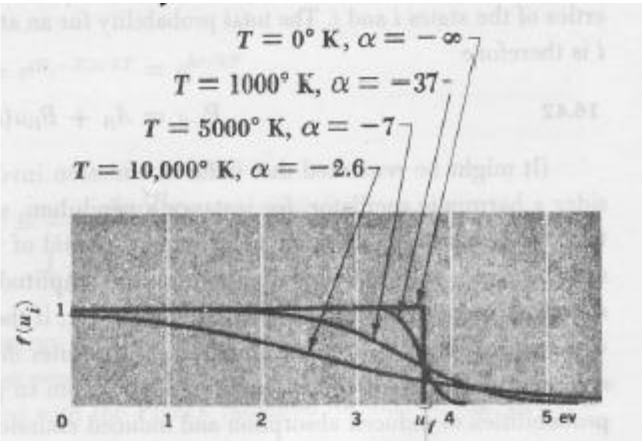
$$\mathcal{N}(E) = \frac{1}{e^{(E-E_F)/k_BT} + 1} \qquad \boxed{=1\ \text{upto}\ E_F}$$

$$D(E) = (2s+1)\frac{m^{3/2}V}{\pi^3\hbar^3\sqrt{2}}\sqrt{E}$$

Fermi Energy



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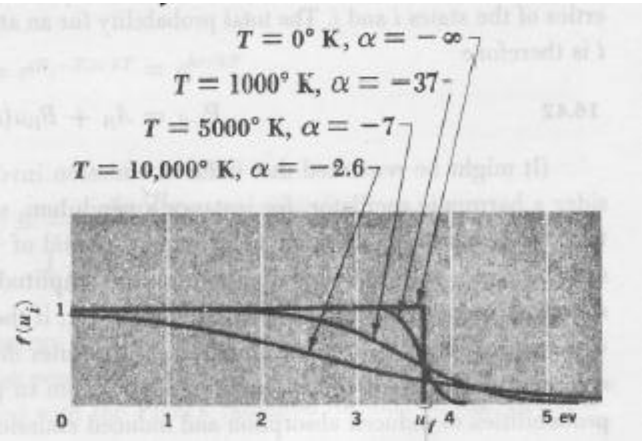
$$N = \int_0^{E_F} \mathcal{N}(E)D(E)dE =$$

Fermi Energy



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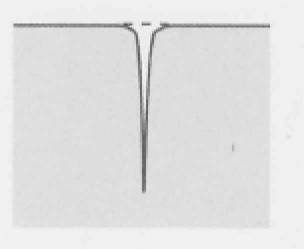
=1 upto E_F

$$N = \int_0^{E_F} \mathcal{N}(E)D(E)dE = \int_0^{E_F} (2s+1)\frac{m^{3/2}V}{\pi^3\hbar^3\sqrt{2}}\sqrt{E}\,dE = (2s+1)\frac{m^{\frac{3}{2}}V}{\pi^3\hbar^3\sqrt{2}}\left(\frac{2}{3}E_F^{\frac{3}{2}}\right)$$

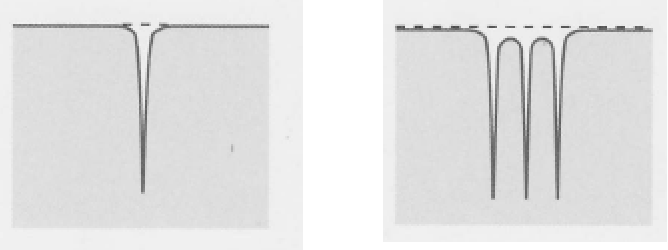
$$E_F = \frac{\pi^2\hbar^2}{m}\left[\frac{3}{(2s+1)\pi\sqrt{2}}\frac{N}{V}\right]^{2/3}$$

For Silver, E_F is 5.5 eV

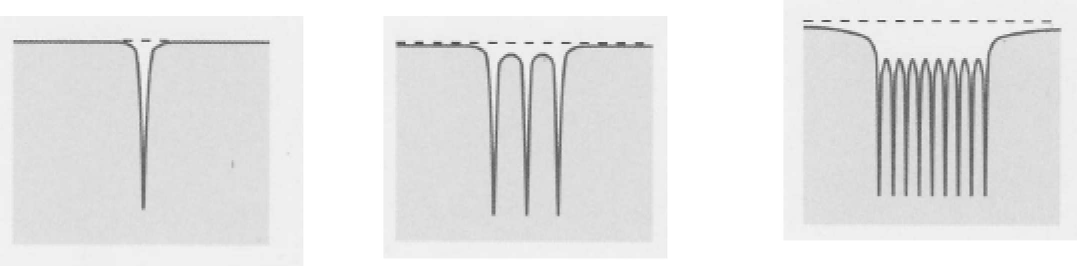
Fermi Gas: Conduction Electrons



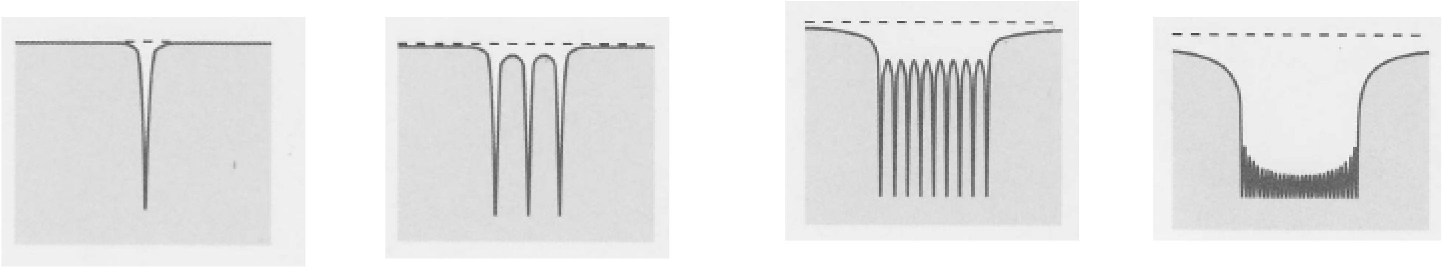
Fermi Gas: Conduction Electrons



Fermi Gas: Conduction Electrons



Fermi Gas: Conduction Electrons



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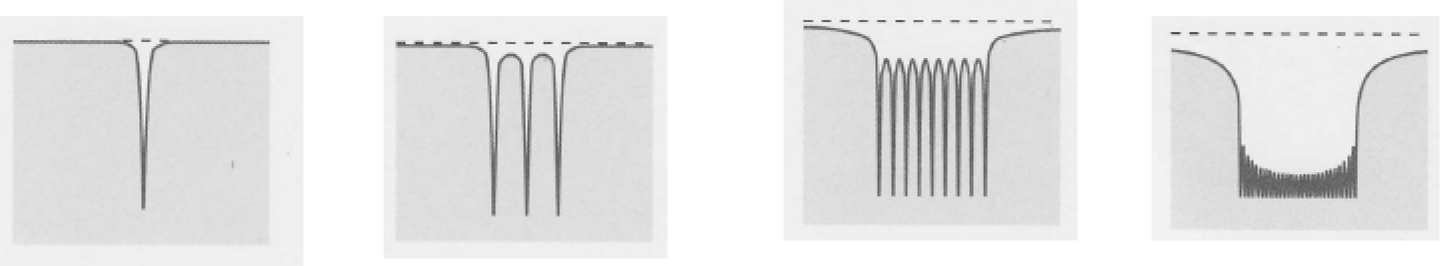
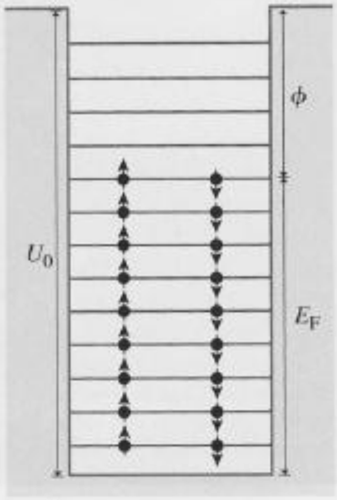


Figure 9.17 Electron energies in a "cold" metal.



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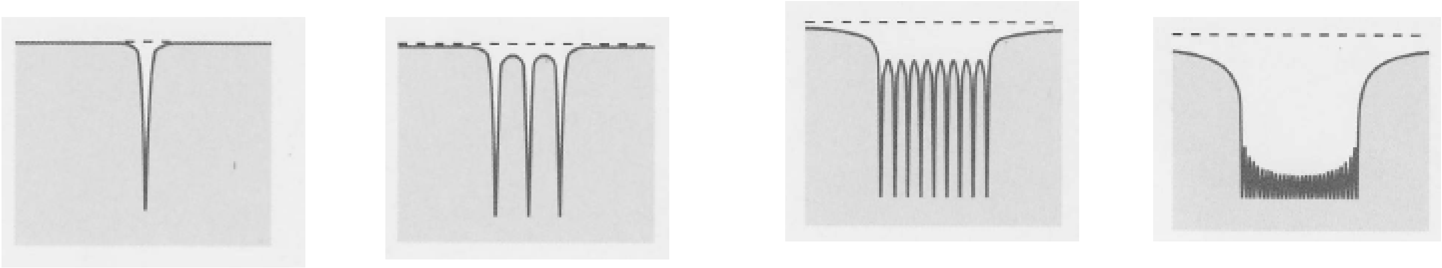
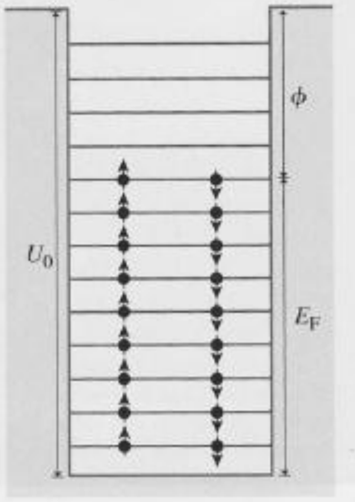


Electrons stack up the energy levels according to Pauli Exclusion Principle.

Figure 9.17 Electron energies in a "cold" metal.



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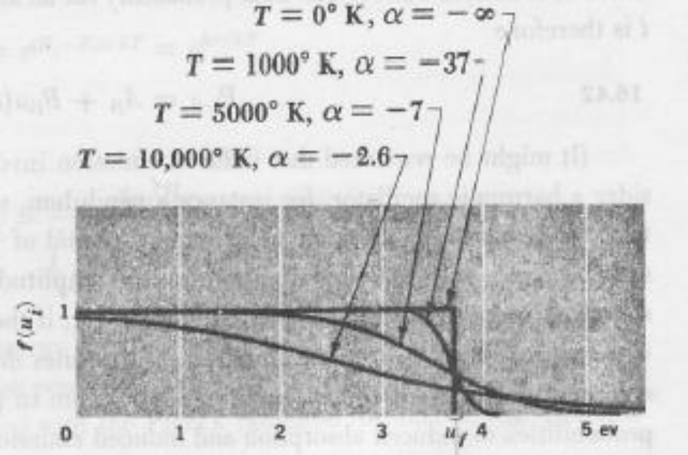

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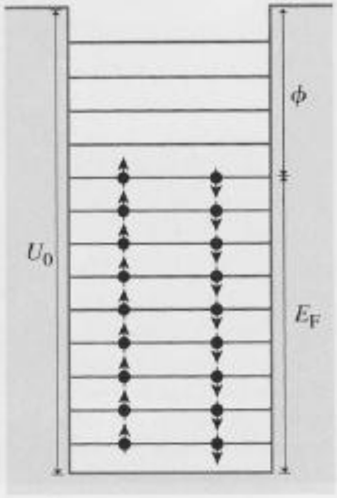
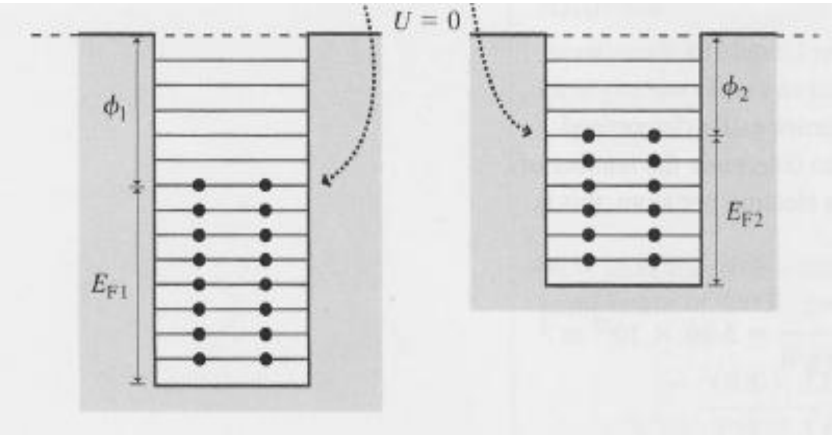


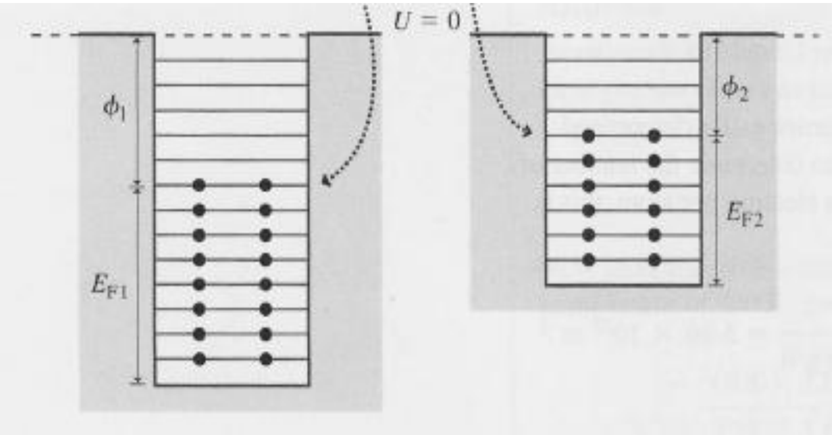
Contact Potential



Work function:

$$\phi = U_0 - E_F$$

Contact Potential



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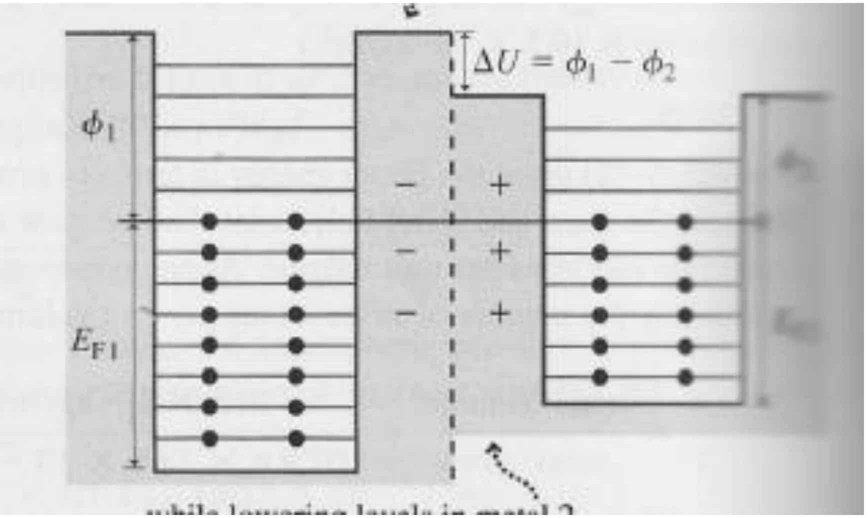
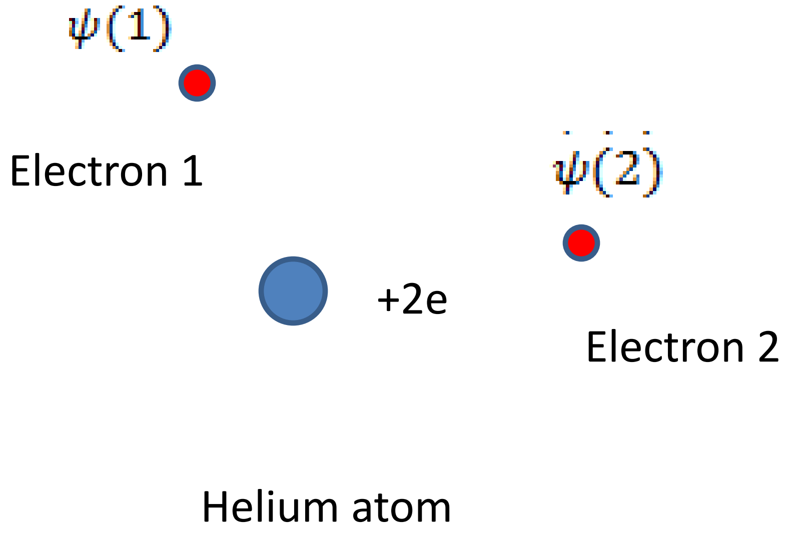


SuperFluid: He_4



$\psi(1)$

Electron 1

$\psi(2)$

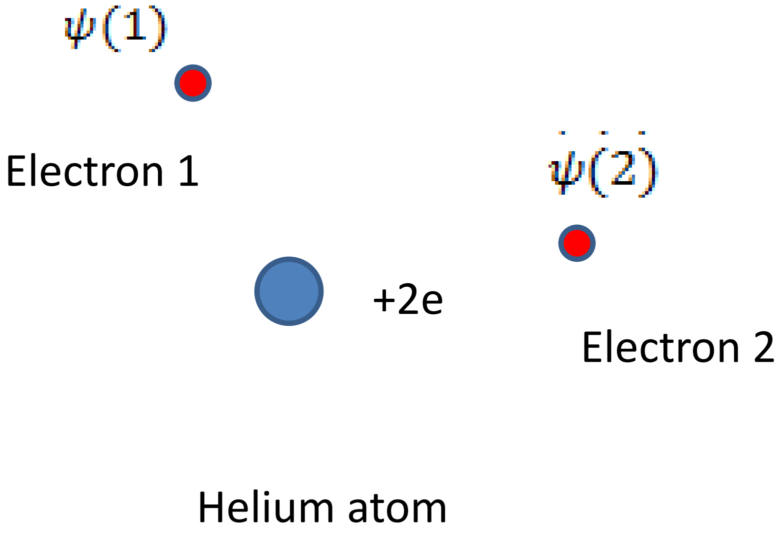
+2e

Electron 2

Helium atom

SuperFluid: He$_4$

Is He a Boson or a Fermion?



$\psi(1)$

Electron 1

$\psi(2)$

+2e

Electron 2

Helium atom

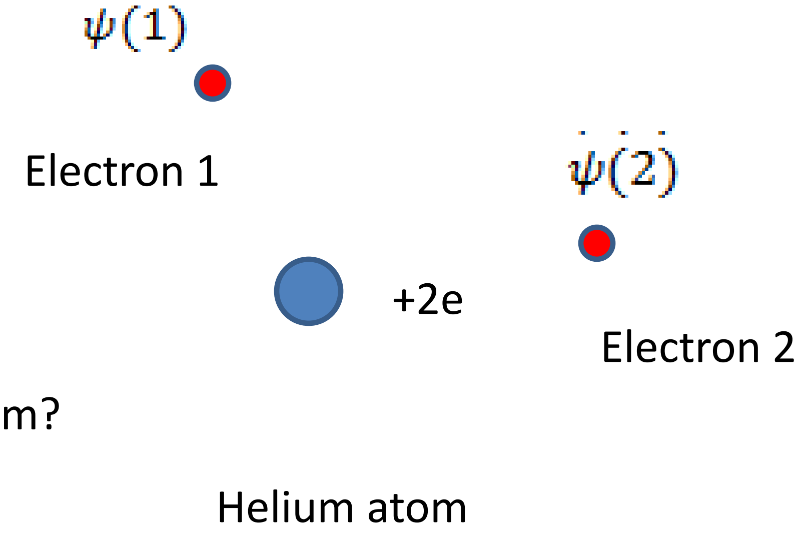
SuperFluid: He_4

He_4 has two electrons of spin 1/2

$S_1 = ½$ $S_2 = ½$

What is the total spin angular momentum?

What are m_s possibilities?



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$\psi(2)$

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Electron 2

Helium atom

SuperFluid: He_4

He_4 has two electrons of spin 1/2

$S_1 = ½$ $S_2 = ½$

What is the total spin angular momentum?

S = ½ + ½ =1   OR S= ½ - ½ =0

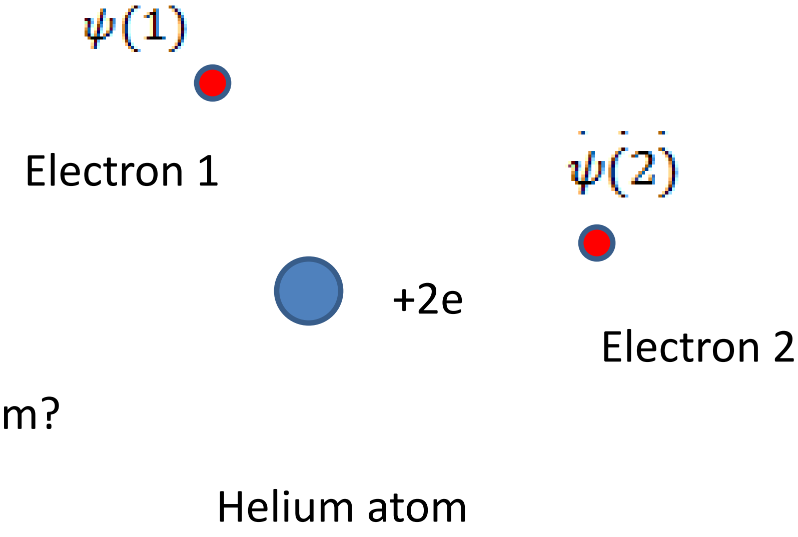
What are m_s possibilities?

For S = 0, m_s =0

For S = 1, m_s =1

m_s =0

m_s =-1



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Electron 1

$\psi(2)$

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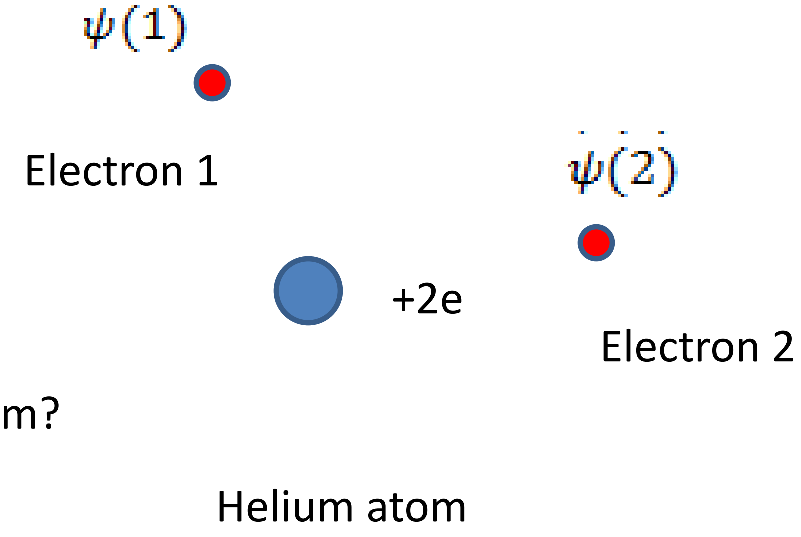
Helium atom

SuperFluid: He_4



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For S = 0, m_s =0 $\uparrow\downarrow \; - \; \downarrow\uparrow$

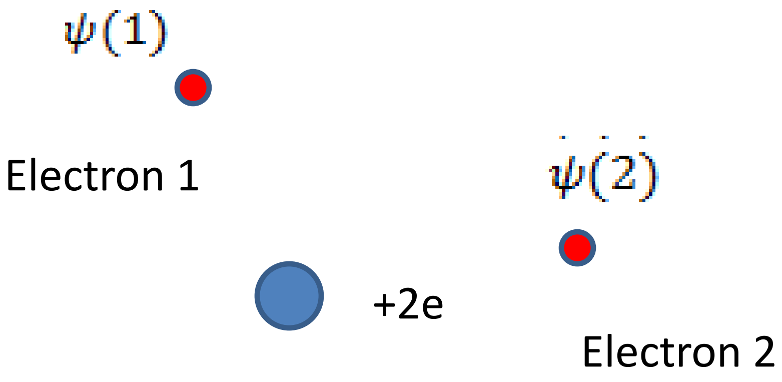
For S = 1, m_s =1 $\uparrow\uparrow$

m_s =0 $\uparrow\downarrow \; + \; \uparrow\downarrow$

m_s =-1 $\downarrow\downarrow$

SuperFluid: He_4

He_4 has two electrons of spin 1/2



Helium atom

$S_1 = ½$ $S_2 = ½$

What is the total spin angular momentum?
S = ½ + ½ =1   OR S= ½ - ½ =0

What are m_s possibilities?

For S = 0, m_s =0   ↑↓ - ↓↑

Singlet anti-symmetric

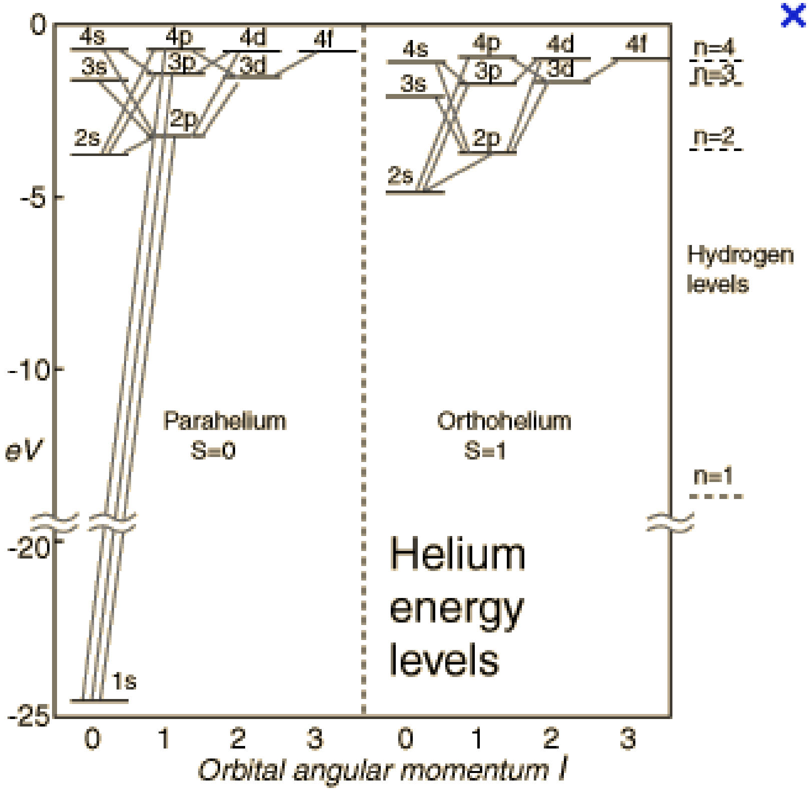
For S = 1, m_s =1   ↑↑

m_s =0   ↑↓ + ↑↓

m_s =-1   ↓↓

Triplet symmetric

Superfluid: He4

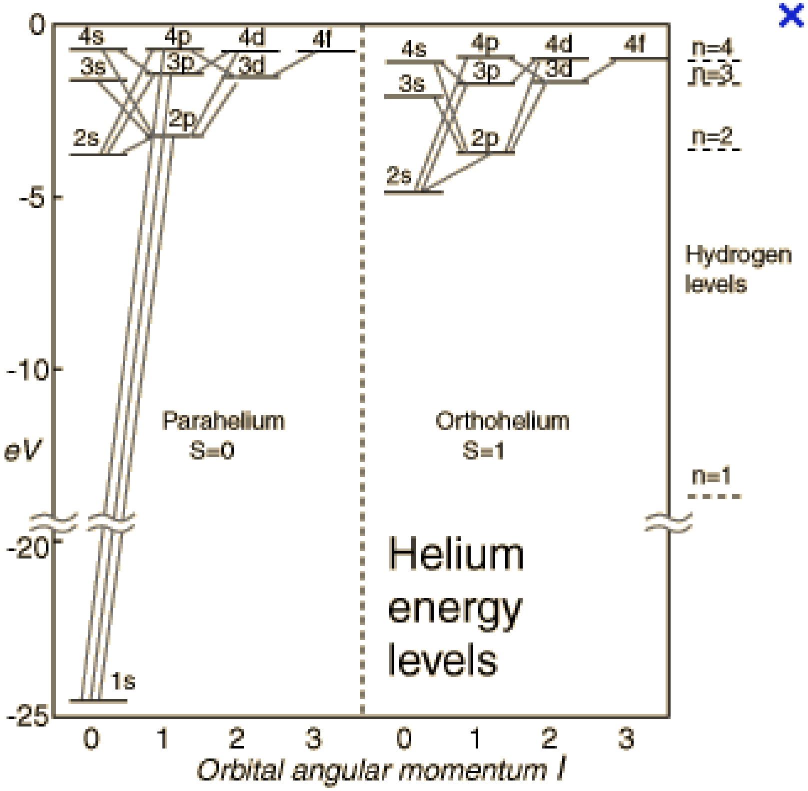


Helium energy levels

- The Spin O state is more stable for the 1S orbital.
- Since electrons in He4 are considered bosons, they can occupy the same energy state.
- As He_4 cools, more stable energy states are sought by electrons.
- Around 2.2K, most electrons in He4 occupies the lowest possible energy states, meaning sharing the same wave function.

$$\frac{1}{\sqrt{2}}\begin{vmatrix} \psi_{1s}(x_1)\uparrow_1 & \psi_{1s}(x_1)\downarrow_1 \\ \psi_{1s}(x_2)\uparrow_2 & \psi_{1s}(x_2)\downarrow_2 \end{vmatrix}$$

Superfluid: He_4



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- In a system of N He atoms:

$$\sum_{j=1}^{N}\underline{e^{-i\overrightarrow{k_j}\cdot\overrightarrow{x_j}}}\begin{vmatrix}\psi_{1s}(x_1)\uparrow_1 & \psi_{1s}(x_1)\downarrow_1\\ \psi_{1s}(x_2)\uparrow_2 & \psi_{1s}(x_2)\downarrow_2\end{vmatrix}$$

Power of coherent waves

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