PH 102: Interactive Lecture 2 Topics

- Particle in a 1-D infinite potential well
- Particle in a 3-D infinite potential well
  - Schrodinger Equation
  - Separation of Variables
  - Energy quantization
  - Wave functions
  - Energy degeneracy
  - Energy split
- 3D Schrodinger Equation for Hydrogen atom
  - Separation of variables
  - Three equations

Schrodinger Equation

Hamiltonian operator (H)

$$H\Psi(x,t) = E\Psi(x,t)$$

Since H = Total Energy = Kinetic energy (T) + Potential energy (U)

$$\left(\frac{p^2}{2m} + U\right)\Psi(x,t) = E\Psi(x,t)$$

$$p = \frac{\hbar}{i}\frac{\partial}{\partial x} \quad \text{and} \quad \mathrm{E} = i\hbar\,\frac{\partial}{\partial t}$$

Time-Dependent Schrodinger Equation

$$\frac{-\hbar^2}{2m}\,\frac{\partial^2\Psi\,(x,t)}{\partial x^2} + U(x)\Psi(x,t) = i\hbar\,\frac{\partial\,\Psi(x,t)}{\partial t}$$

Schrodinger Equation

Time-Dependent Schrodinger Equation

$$\frac{-\hbar^2}{2m}\frac{\partial^2 \Psi(x,t)}{\partial x^2} + U(x)\Psi(x,t) = i\hbar\frac{\partial \Psi(x,t)}{\partial t}$$

Separation of Variables $\Psi(x,t) = \psi(x)\ \phi(t)$

Spatial part of $\Psi(x,t)$: $\frac{-\hbar^2}{2m}\frac{1}{\psi(x)}\frac{\partial^2 \psi(x)}{\partial x^2} + U(x) = \mathrm{C}$

Temporal part of $\Psi(x,t)$: $i\hbar\ \frac{1}{\phi(t)}\frac{\partial \phi(t)}{\partial t} = \mathrm{C}$

time-independent Schrodinger Equation:

$$\frac{-\hbar^2}{2m}\frac{\partial^2 \psi(x)}{\partial x^2} + U(x)\,\psi(x) = \mathrm{E}\,\psi(x)$$

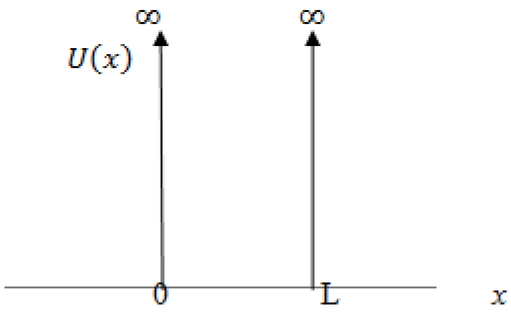
Quantum Problem Solving Schema

- Think about potential energy and Hamiltonian
- Divide regions, if possible
- Choose appropriate coordinates
- Write Schrodinger Equation for each region
- Separation of variables, if possible
- Set "constant"s if applicable
- Solve for wave function
  - Boundary conditions
  - Normalization
- Solve for energy
- Build energy level diagrams and inspect for energy degeneracies

1-D infinite potential well

$$\frac{-\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + U(x)\,\psi(x) = \mathrm{E}\,\psi(x)$$

$$U(x) = \begin{cases} \infty & x \le 0 \\ 0 & 0 < x < L \\ \infty & x \ge L \end{cases}$$



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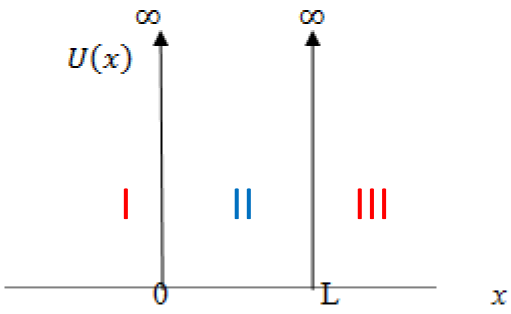
→ Region I (for $x \le 0$)

→ Region II (for $0 < x < L$)

→ Region III (for $x \ge L$)



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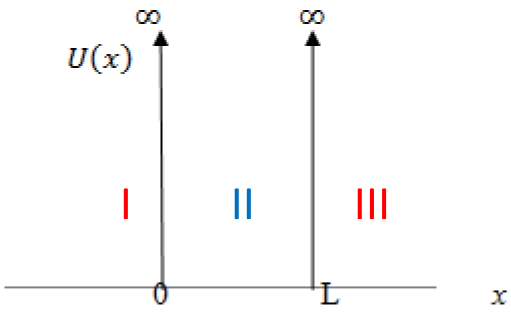
→ Region II

→ Region III



Regions I and III, no wave function can exist → $\psi_{x \le 0}(x) = 0$

$\psi_{x \ge L}(x) = 0$

1-D infinite potential well

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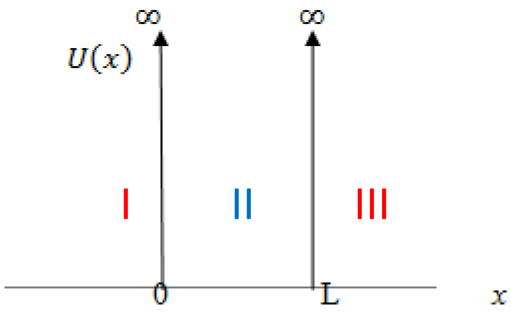
→ Region III



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1-D infinite potential well

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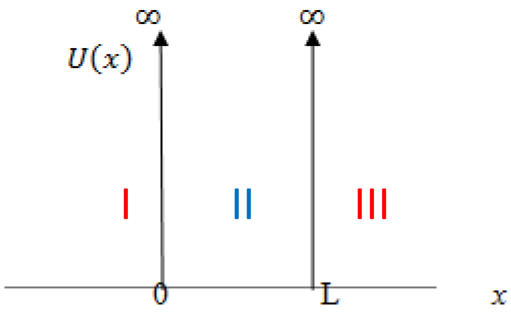
$x \le 0$ → Region I
$0 < x < L$ → Region II
$x \ge L$ → Region III



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1-D infinite potential well

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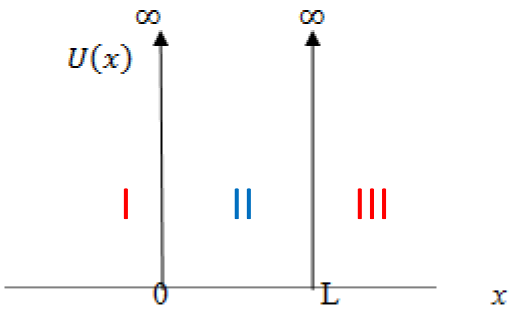

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$\psi_{x\ge L}(x = 0) = 0$ → $\psi_{0<x<L}(x = L) = \sin(kL) = 0$

1-D infinite potential well

$$\frac{-\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + U(x)\,\psi(x) = \mathrm{E}\,\psi(x)$$

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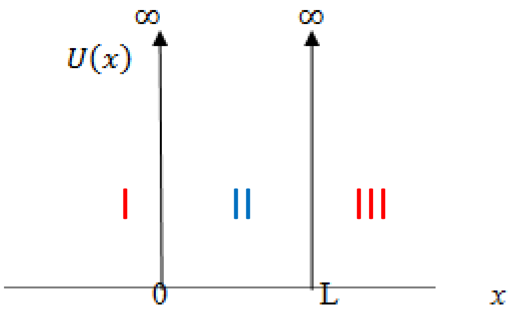

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Region II, $\frac{-\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} = \mathrm{E}\,\psi(x)$ → $\psi_{0<x<L}(x) = A\sin(kx)$; $k = \sqrt{\frac{2mE}{\hbar^2}}$

$\psi_{x\ge L}(x=0) = 0$ → $\psi_{0<x<L}(x=L) = \sin(kL) = 0$

$$kL = \sqrt{\frac{2mE}{\hbar^2}}L = n\pi$$ where n =1, 2, 3, etc.

$$E = \frac{n^2\pi^2\hbar^2}{2mL^2}$$ Energy quantization

1-D infinite potential well

Normalization:

$$\psi_{0<x<L}(x) = A\sin(kx) = \mathrm{A}\sin(\frac{n\pi x}{L})$$

$$\int_0^L |\psi_{0<x<L}(x)|^2 = 1 = \mathrm{A}^2 \int_0^L \sin\left(\frac{n\pi x}{L}\right)^2 dx =$$

$$sin^2x = \frac{1 - cos2x}{2}$$

$$cos^2x = \frac{1 + cos2x}{2}$$

1-D infinite potential well

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$$sin^2 x = \frac{1 - cos2x}{2}$$

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1-D infinite potential well

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Wave function: $\psi_{0<x<L}(x) = \sqrt{\frac{2}{L}}\ \sin(\frac{n\pi x}{L})$

Energy $E = \frac{n^2\,\pi^2\,\hbar^2}{2mL^2}$

1-D infinite potential well

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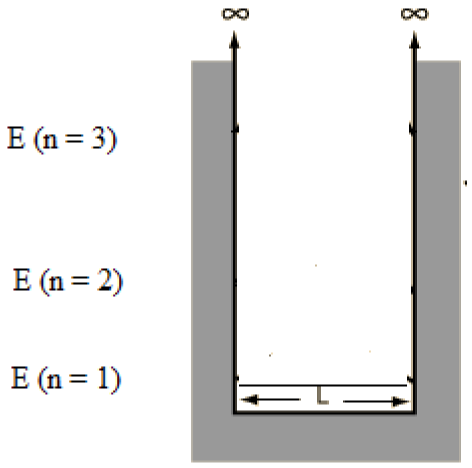
Energy $E = \frac{n^2\,\pi^2\,\hbar^2}{2mL^2}$

Obtain the first three energy levels ($\frac{\pi^2\,\hbar^2}{2mL^2}$) and draw their associated wave functions



1-D infinite potential well

Normalization:

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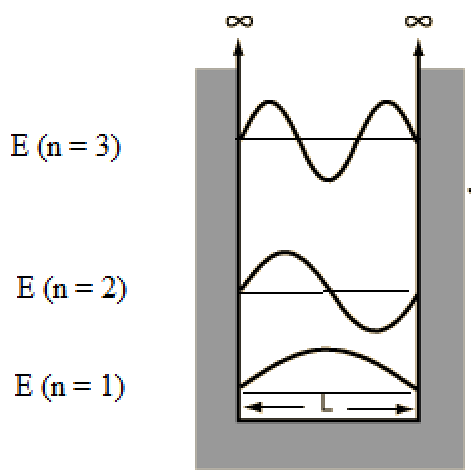
Wave function: $\psi_{0<x<L}(x) = \sqrt{\frac{2}{L}}\sin(\frac{n\pi x}{L})$

Energy $E = \frac{n^2\pi^2\hbar^2}{2mL^2}$

$$E\ (n=3) = 9\ \frac{\pi^2\hbar^2}{2mL^2}$$

$$E\ (n=2) = 4\ \frac{\pi^2\hbar^2}{2mL^2}$$

$$E\ (n=1) = 1\ \frac{\pi^2\hbar^2}{2mL^2}$$

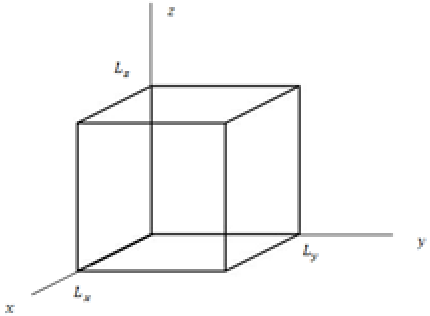


Particle in a 3-d infinite well

$$\frac{-\hbar^2}{2m}\nabla^2\psi(\vec{x}) + U(\vec{x})\psi(\vec{x}) = E\,\psi(\vec{x})$$

In (x, y, z) coordinates, $\vec{x} = (x, y, z)$, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi(x,y,z) = -\frac{2m}{\hbar^2}(E - U(x,y,z))\psi(x,y,z)$$



$$U(\vec{x}) = \begin{cases} 0 & 0 < x < L_x, 0 < y < L_y . 0 < z < L_z \\ \infty & otherwise \end{cases}$$

Wave functions exist only inside the 3-d infinite well.

Particle in a 3-d infinite well

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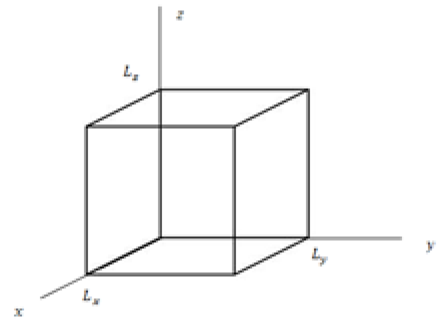

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Separation of variables: $\psi(\vec{x}) = \psi(x, y, z) = F(x)G(y)H(z)$

Particle in a 3-d infinite well

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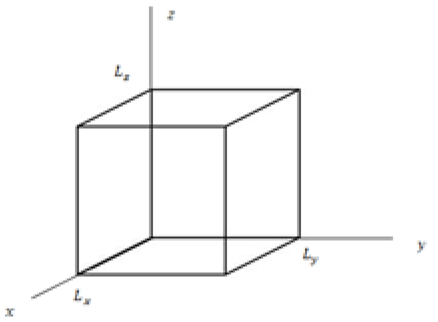

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Wave functions exist only inside the 3-d infinite well.

Separation of variables: $\psi(\vec{x}) = \psi(x, y, z) = F(x)G(y)H(z)$

$$\frac{1}{F(x)}\frac{\partial^2 F(x)}{\partial x^2} + \frac{1}{G(y)}\frac{\partial^2 G(y))}{\partial y^2} + \frac{1}{H(z)}\frac{\partial^2 H(z)}{\partial z^2} = -\frac{2mE}{\hbar^2}$$

Particle in a 3-d infinite well

$$\frac{-\hbar^2}{2m}\nabla^2\psi(\vec{x}) + U(\vec{x})\psi(\vec{x}) = E\,\psi(\vec{x})$$

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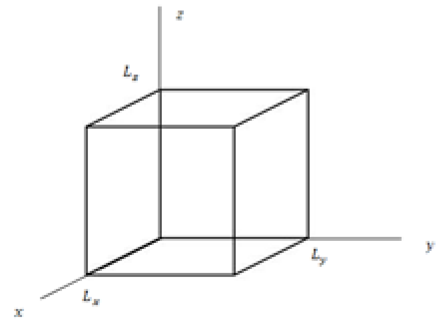

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=constant

constant=Cx **Constant=Cy** **Constant=Cz**

$$Cx + Cy + Cz = -\frac{2mE}{\hbar^2}$$

Particle in a 3-d infinite well

1-D solutions:

$$\frac{\partial^2 \psi(x)}{\partial x^2} = \frac{-2m\,E}{\hbar^2}\,\psi(x)$$

Wave function: $\psi_{0<x<L}(x) = \sqrt{\frac{2}{L}}\,\sin(\frac{n\pi x}{L})$

Energy $E = \frac{n^2\,\pi^2\,\hbar^2}{2mL^2}$

$$\begin{cases} \frac{d\,F(x)}{dx^2} = C_x F(x) \\ \frac{d\,G(y)}{dy^2} = C_y G(y) \\ \frac{d\,H(z)}{dz^2} = C_z H(z) \end{cases}$$

Particle in a 3-d infinite well

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Energy $E = \frac{n^2\,\pi^2\,\hbar^2}{2mL^2}$

3-D solutions: wave functions

$$\begin{cases} \frac{d\,F(x)}{dx^2} = C_x F(x) & \rightarrow F(x) = A_x \sin\frac{n_x \pi x}{L_x} \\ \frac{d\,G(y)}{dy^2} = C_y G(y) & \rightarrow G(y) = A_y \sin\frac{n_y \pi y}{L_y} \\ \frac{d\,H(z)}{dz^2} = C_z H(z) & \rightarrow H(z) = A_z \sin\frac{n_z \pi z}{L_z} \end{cases}$$

Particle in a 3-d infinite well

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$Cx = -\frac{n_x^2 \pi^2}{L_x{}^2}$ $\quad Cy = -\frac{n_y^2 \pi^2}{L_y{}^2}$ $\quad Cz = -\frac{n_z^2 \pi^2}{L_z{}^2}$

$$Cx + Cy + Cz = -\frac{2mE}{\hbar^2} = -\frac{n_x^2 \pi^2}{L_x{}^2} - \frac{n_y^2 \pi^2}{L_y{}^2} - \frac{n_z^2 \pi^2}{L_z{}^2}$$

Particle in a 3-D infinite well

$$E_{(n_x,n_y,n_z)} = \left(\frac{n_x^2}{L_x{}^2} + \frac{n_y^2}{L_y{}^2} + \frac{n_z^2}{L_z{}^2}\right)\frac{\pi^2\hbar^2}{2m}$$

$$\psi\,(x,y,z) = F(x)G(y)H(z) = A\,\sin\frac{n_x\pi x}{L_x}\sin\frac{n_y\pi y}{L_y}\sin\frac{n_z\pi z}{L_z}$$

Lowest energy state =

Wave function for the lowest energy state =

Particle in a 3-D infinite well

$$E_{(n_x,n_y,n_z)} = \left(\frac{n_x^2}{{L_x}^2} + \frac{n_y^2}{{L_y}^2} + \frac{n_z^2}{{L_z}^2}\right)\frac{\pi^2\hbar^2}{2m}$$

$$\psi\,(x,y,z) = F(x)G(y)H(z) = A\,\sin\frac{n_x\pi x}{L_x}\sin\frac{n_y\pi y}{L_y}\sin\frac{n_z\pi z}{L_z}$$

Lowest energy state = $E_{(1,1,1)} = \left(\frac{1^2}{{L_x}^2} + \frac{1^2}{{L_y}^2} + \frac{1^2}{{L_z}^2}\right)\frac{\pi^2\hbar^2}{2m}$

Wave function for the lowest energy state = $\psi_{(1,1,1)} = A\,\sin\frac{\pi x}{L_x}\sin\frac{\pi y}{L_y}\sin\frac{\pi z}{L_z}$

Particle in a 3-D infinite well

$$E_{(n_x,n_y,n_z)} = \left(\frac{n_x^2}{L_x{}^2} + \frac{n_y^2}{L_y{}^2} + \frac{n_z^2}{L_z{}^2}\right)\frac{\pi^2\hbar^2}{2m}$$

$$\psi\,(x,y,z) = F(x)G(y)H(z) = A\,\sin\frac{n_x\pi x}{L_x}\sin\frac{n_y\pi y}{L_y}\sin\frac{n_z\pi z}{L_z}$$

When $L_x = L_y = L_z = L$

Lowest energy state =

Wave function for the lowest energy state =

Second lowest energy state(s) =

Wave functions for the second lowest energy state(s) =

Particle in a 3-D infinite well

$$E_{(n_x,n_y,n_z)} = \left(\frac{n_x^2}{L_x{}^2} + \frac{n_y^2}{L_y{}^2} + \frac{n_z^2}{L_z{}^2}\right)\frac{\pi^2\hbar^2}{2m}$$

$$\psi(x,y,z) = F(x)G(y)H(z) = A\ \sin\frac{n_x\pi x}{L_x}\sin\frac{n_y\pi y}{L_y}\sin\frac{n_z\pi z}{L_z}$$

When $L_x = L_y = L_z = L$

Lowest energy state = $E = 3\left(\frac{\pi^2\hbar^2}{2mL^2}\right)$

Wave function for the lowest energy state = $\psi_{(1,1,1)} = A\ \sin\frac{\pi x}{L}\sin\frac{\pi y}{L}\sin\frac{\pi z}{L}$

Second lowest energy state(s) = $E_{(2,1,1)} = E_{(1,2,1)} = E_{(1,1,2)} = (2^2 + 1^2 + 1^2)\left(\frac{\pi^2\hbar^2}{2mL^2}\right) = 6\left(\frac{\pi^2\hbar^2}{2mL^2}\right)$

Wave functions for the second lowest energy state(s) = $\begin{cases}\psi_{(2,1,1)} = A\ \sin\frac{2\pi x}{L}\sin\frac{\pi y}{L}\sin\frac{\pi z}{L}\\ \psi_{(1,2,1)} = A\ \sin\frac{\pi x}{L}\sin\frac{2\pi y}{L}\sin\frac{\pi z}{L}\\ \psi_{(1,1,2)} = A\ \sin\frac{\pi x}{L}\sin\frac{\pi y}{L}\sin\frac{2\pi z}{L}\end{cases}$

An electron in a cubit 3d infinite well of 1 nm at the E(2,1,1) state

$$E_{(2,1,1)} = 6\left(\frac{\pi^2\hbar^2}{2mL^2}\right) = (2^2 + 1^2 + 1^2)\frac{\pi^2(1.055 \text{ x } 10^{-34}\text{ J sec})^2}{2(9.11 \text{ x } 10^{-31}\text{kg})(10^{-9}\text{ m})^2}$$
$$= 3.62 \text{ x } 10^{-19}\text{ J} = 2.26\text{ eV (the same as } E_{(1,2,1)} = E_{(1,1,2)})$$

$$\text{Where}\begin{cases}\text{electron mass} = 9.11 \text{ x } 10^{-31}\text{kg} \\ h = 1.055 \text{ x } 10^{-34}\text{ J sec} \\ L = 10^{-9}\text{ m}\end{cases} \quad \text{and } 1\text{ eV} = 1.602 \text{ x } 10^{-19}\text{ J}$$

Where is a particle with the value most likely to be found?

Probability density

- $|\psi_{(2,1,1)}|^2 = A^2\left(\sin\frac{2\pi x}{L}\right)^2\left(\sin\frac{\pi y}{L}\right)^2\left(\sin\frac{\pi z}{L}\right)^2$

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Since the value of $(sin\theta)^2$ *is highest when* $\theta = \frac{1}{2}\pi, \frac{3}{2}\pi, etc.$, the probability

$$\begin{cases}x = \frac{L}{4}, \frac{3L}{4}\\ y = \frac{L}{2}\\ z = \frac{L}{2}\end{cases}$$

Energy Split

Consider (that is, a slightly non-symmetric box along the z axis)

$$L_x = L_y = L_z = L \longrightarrow L_x = L_y = L,\ L_z = .9\,L$$

$$E_{(1,1,1)} = (2^2 + 1^2 + 1^2)\left(\frac{\pi^2\hbar^2}{2mL^2}\right) = 3\ (\frac{\pi^2\hbar^2}{2mL^2})$$

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$$E_{(2,1,1)} = E_{(1,2,1)} = E_{(1,1,2)} = (2^2 + 1^2 + 1^2)\left(\frac{\pi^2\hbar^2}{2mL^2}\right) = 6\left(\frac{\pi^2\hbar^2}{2mL^2}\right)$$

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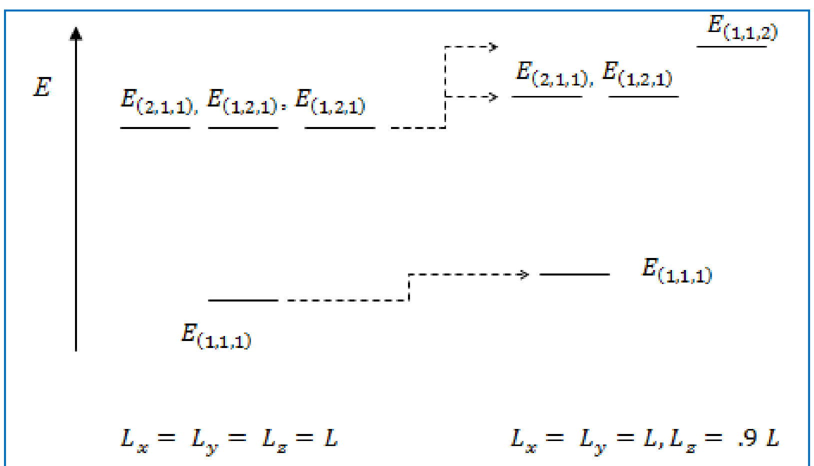

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Hydrogen atom

- Potential created by Coulomb interactions between electron ($-e$) and proton ($+e$)

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$$U(x) = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}$$

Hydrogen atom

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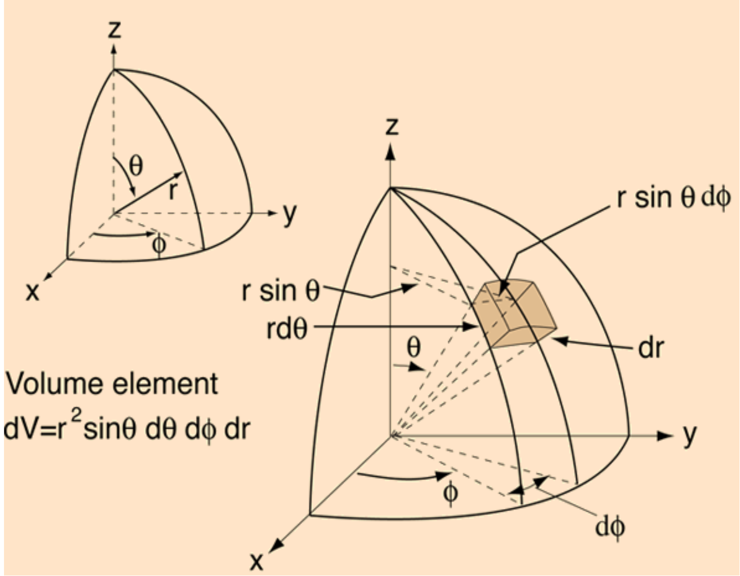
$$U(x) = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}$$

  - Symmetric in r
  - Choose (r, θ, ϕ)

$$(x, y, z) \quad \leftrightarrow \quad (r, \theta, \phi)$$

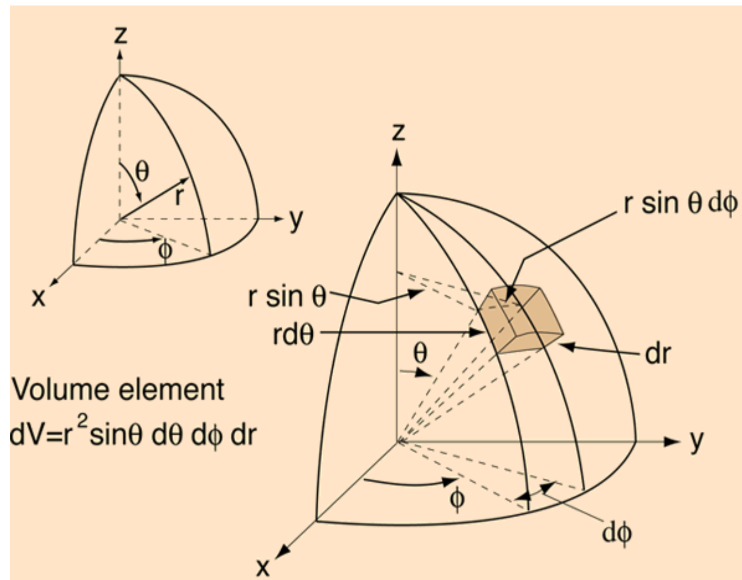


$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$(x, y, z) \quad \leftrightarrow \quad (r, \theta, \phi)$$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \phi = tan^{-1}\dfrac{y}{x} \\ \theta = cos^{-1}\dfrac{z}{r} \end{cases}$$

$$\begin{cases} x = r sin\theta cos\phi \\ y = r\, sin\theta sin\,\phi \\ z = r cos\theta \end{cases}$$



$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla^2 = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + csc\theta\,\frac{\partial}{\partial\theta}\left(sin\theta\,\frac{\partial}{\partial\theta}\right) + csc^2\theta\,\frac{\partial}{\partial\phi^2}\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\,\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial}{\partial\phi^2}$$

Schrodinger Equation

$$\frac{-\hbar^2}{2m} \nabla^2 \psi\left(\vec{x}\right) + U(\vec{x})\psi\left(\vec{x}\right) = E\,\psi(\vec{x})$$

$$\nabla^2 = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + csc\theta\,\frac{\partial}{\partial \theta}\left(sin\theta\,\frac{\partial}{\partial \theta}\right) + csc^2\theta\,\frac{\partial}{\partial \phi^2}\right]$$

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$$csc\theta\,\frac{\partial}{\partial\theta}\left(sin\theta\,\frac{\partial}{\partial\theta}\right)\psi(r,\theta,\phi) + csc^2\theta\,\frac{\partial^2}{\partial\phi^2}\psi(r,\theta,\phi)$$

$$= \left[-\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{2mr^2}{\hbar^2}(E - U(r))\right]\psi(r,\theta,\phi)$$

Schrodinger Equation

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$$= [-\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{2mr^2}{\hbar^2}\,(E - U(r))]\,\psi(r,\theta,\phi)$$

Separation of variables

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$$

$$\frac{\partial\psi}{\partial r} = \Theta\Phi\frac{\partial R}{\partial r}$$
$$\frac{\partial\psi}{\partial\theta} = R\Phi\frac{\partial\Theta}{\partial\theta}$$
$$\frac{\partial^2\Phi}{\partial\phi^2} = R\Theta\frac{\partial^2\Phi}{\partial\phi^2}$$

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$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$$

$$\frac{\partial\psi}{\partial r} = \Theta\Phi\frac{\partial R}{\partial r}$$

$$\frac{\partial\psi}{\partial\theta} = R\Phi\frac{\partial\Theta}{\partial\theta}$$

$$\frac{\partial^2\Phi}{\partial\phi^2} = R\Theta\frac{\partial^2\Phi}{\partial\phi^2}$$

$$R\Phi\,csc\theta\,\frac{\partial}{\partial\theta}\left(sin\theta\,\frac{\partial\Theta}{\partial\theta}\right) + R\Theta csc^2\theta\,\frac{\partial^2\Phi}{\partial\phi^2} = -\Theta\Phi\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - \frac{2mr^2}{\hbar^2}(E - U(r))\,R\Theta\Phi$$

Schrodinger Equation

$$\frac{1}{\Theta}\ csc\theta\ \frac{\partial}{\partial\theta}\left(sin\theta\,\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Phi}csc^{2}\theta\,\frac{\partial^{2}\Phi}{\partial\phi^{2}}=\ -\frac{1}{R}\frac{\partial}{\partial r}\left(r^{2}\frac{\partial R}{\partial r}\right)-\frac{2mr^{2}}{\hbar^{2}}\ (E-U(r))=C\ (\text{Constant})$$

Angular part

Radial part

Schrodinger Equation

$$\frac{1}{\Theta}\ csc\theta\ \frac{\partial}{\partial\theta}\left(sin\theta\,\frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\Phi}csc^2\theta\,\frac{\partial^2\Phi}{\partial\phi^2} = -\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - \frac{2mr^2}{\hbar^2}\left(E - U(r)\right) = C\ (\text{Constant})$$

Angular part

Radial part

-l (l + 1)

Schrodinger Equation

$$\frac{1}{\Theta}\ csc\theta\ \frac{\partial}{\partial\theta}\left(sin\theta\,\frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\Phi}csc^2\theta\,\frac{\partial^2\Phi}{\partial\phi^2} = -\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - \frac{2mr^2}{\hbar^2}\left(E - U(r)\right) = C\ (\text{Constant})$$

Angular part

-*l* (*l* + 1)

Radial part

-*l* (*l* + 1)

-*l* (*l* + 1)

Schrodinger Equation

$$\frac{1}{\Theta}\csc\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Phi}\csc^2\theta\frac{\partial^2\Phi}{\partial\phi^2} = -\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)-\frac{2mr^2}{\hbar^2}\left(E-U(r)\right)=C\ (\text{Constant})$$

Angular part → $-l(l+1)$

Radial part → $-l(l+1)$

Constant → $-l(l+1)$

$$\begin{cases} -\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)-\frac{2mr^2}{\hbar^2}\left(E-U(r)\right)=C=-l(l+1) \\ \frac{1}{\Theta}\csc\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\Phi}\csc^2\theta\frac{\partial^2\Phi}{\partial\phi^2}=C=-l(l+1) \end{cases}$$

Schrodinger Equation

$$\frac{1}{\Theta}\, csc\theta\, \frac{\partial}{\partial\theta}\left(sin\theta\, \frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\Phi} csc^2\theta\, \frac{\partial^2\Phi}{\partial\phi^2} = -\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - \frac{2mr^2}{\hbar^2}\,(E - U(r)) = C \text{ (Constant)}$$

Angular part → $-l\,(l + 1)$

Radial part → $-l\,(l + 1)$

Constant → $-l\,(l + 1)$

$$\begin{cases} -\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - \frac{2mr^2}{\hbar^2}\,(E - U(r)) = C = -l(l+1) \\ \frac{1}{\Theta}\, csc\theta\, \frac{\partial}{\partial\theta}\left(sin\theta\, \frac{\partial\Theta}{\partial\theta}\right) + \frac{1}{\Phi} csc^2\theta\, \frac{\partial^2\Phi}{\partial\phi^2} = C = -l(l+1) \end{cases}$$

$$\frac{1}{\Theta}\, sin\theta\, \frac{\partial}{\partial\theta}\left(sin\theta\, \frac{\partial\Theta}{\partial\theta}\right) + l(l+1)sin^2\theta = -\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2}$$

$$= m_l^2 \text{ (another constant)}$$

Schrodinger Equation

$$\begin{cases} \dfrac{\partial^2 \Phi}{\partial \phi^2} = -m_l^2 \Phi \\ sin\theta \dfrac{\partial}{\partial \theta}\left(sin\theta \dfrac{\partial \Theta}{\partial \theta}\right) + [l(l+1)sin^2\theta - m_l^2]\Theta = 0 \\ \dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial R}{\partial r}\right) + \dfrac{2mr^2}{\hbar^2}\left(E - U(r)\right)R - l(l+1)R = 0 \end{cases}$$

Schrodinger Equation

$$\begin{cases} \frac{\partial^2 \Phi}{\partial \phi^2} = -m_l^2 \Phi & \text{Azimuthal Equation} \\ sin\theta \frac{\partial}{\partial \theta}\left(sin\theta \frac{\partial \Theta}{\partial \theta}\right) + [l(l+1)sin^2\theta - m_l^2]\Theta = 0 & \text{Polar Equation} \\ \frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{2mr^2}{\hbar^2}\left(E - U(r)\right)R - l(l+1)R = 0 & \text{Radial Equation} \end{cases}$$

Quantum Problem Solving Schema

- Think about potential energy and Hamiltonian
- Divide regions, if possible
- Choose appropriate coordinates
- Write Schrodinger Equation for each region
- Separation of variables, if possible
- Set “constant”s if applicable
- Solve for wave function
  - Boundary conditions
  - Normalization
- Solve for energy
- Build energy level diagrams and inspect for energy degeneracies